INTRODUCTION
WHAT IS MATHEMATICAL MODELING?

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Mathematical Modeling in a Nutshell

Mathematicians are in the habit of dividing the universe into two parts: mathematics, and everything else, that is, the rest of the world, sometimes called "the real world". People often tend to see the two as independent from one another – nothing could be further from the truth. When you use mathematics to understand a situation in the real world, and then perhaps use it to take action or even to predict the future, both the real-world situation and the ensuing mathematics are taken seriously. The situations and the questions associated with them may be any size from huge to little. The big ones may lead to lifetime careers for those who study them deeply and special curricula or whole university departments may be set up to prepare people for such careers. Electromagnetic theory, medical imaging, and cryptography are some such examples. At the other end of the scale, there are small situations and corresponding questions, although they may be of great importance to the individuals involved: planning a trip, scheduling the preparation of Thanksgiving Dinner, hiring a new assistant, or bidding in an auction. Problems of intelligent citizenship vary greatly in complexity: deciding whether to vote sincerely in the first round of an election, or to vote so as to try to remove the most dangerous threat to your actual favorite candidate; planning the one-way traffic patterns for your downtown; thinking seriously, when the school system argues about testing athletes for steroids, whether you prefer a test that catches almost all the users at the price of designating some non-users as (false) positives, or a test in which almost everybody it catches *is* a user, but misses some of the actual users.

Whether the problem is huge or little, the process of "interaction" between the mathematics and the real world is the same: The real situation usually has so many facets that you can't take everything into account, so you decide which aspects are most important and keep those. At this point, you have an idealized version of the real-world situation, which you can then translate into mathematical terms. What do you have now? A *mathematical model* of the idealized question. You then apply your mathematical instincts and knowledge to the model, and gain interesting insights, examples, approximations, theorems, and algorithms. You translate all this back into the real-world situation, and you hope to have a theory for the idealized question. But you have to check back: Are the results practical, the answers reasonable, the consequences acceptable? If so, great! If not, take another look at the choices you made at the beginning, and try again. This entire process is what is called *mathematical modeling*.

You may be wondering how mathematical modeling differs from what you already teach, particularly, "problem solving". Problem solving may not refer to the outside world at all. Even when it does, problem solving usually begins with the idealized real-world situation in mathematical terms, and ends with a mathematical result. Mathematical modeling, on the other hand, begins in the "unedited" real world, requires problem *formulating* before problem *solving*, and once the problem is solved, moves back into the real world where the results are considered in their original context. Additionally, it would take us too far afield to discuss *whimsical* problems, where mythical kingdoms and incredible professions and procedures may become the setting of some lovely mathematics. They make no pretense of being problems motivated by the real world.

Mathematical Modeling and Education

Now that we have an idea about what mathematical modeling is in the real world, what do we do about it in mathematics education? One hundred years ago, the big areas – classical physics, astronomy, cartography, and surveying, for instance – were taught in university mathematics departments, perhaps called departments of mathematics and astronomy. Nowadays, in the United States at least, these are taught in science or engineering departments. These branches of science are big and they are very old. What about areas that have become major appliers of mathematics during the last century? Information theory and cryptography may be included in the curricula of electrical engineering, inventory control, programming (as in "linear"), scheduling and queuing in operations research, and fair division and voting in political science. These topics are such exciting new areas of application, often of discrete mathematics, that they frequently have a home in mathematics, as well. Who is going to "own" them in the long run is undecided.

What do we, as mathematicians and mathematics educators, conclude? Many scientific disciplines use mathematics in their development and practice, and when they are faithful to the science they do indeed check which aspects of the situation they have kept and which they have chosen to ignore. Engineers and scientists, be they physical, social, or biological, have not expected mathematics to teach the modeling point of view for them within a scientific framework, although preparing for this kind of reasoning is part of mathematics. Since the scientists will do mathematical modeling anyway, can we just leave mathematical modeling to them? *Absolutely not*. Why not? Mathematics education is at the very least responsible for teaching how to use mathematics in everyday life and in intelligent citizenship, and let's not forget it. Actually, any separation of science from everyday life is a delusion. Both everyday life and intelligent citizenship often also involve scientific issues. So what really matters in mathematics education is learning and practicing the mathematical modeling process. The particular field of application, whether it is everyday life or being a good citizen or understanding some piece of science, is less important than the experience with this thinking process.

Mathematical Modeling in School

Let us now look at mathematical modeling as an essential component of school mathematics. How successfully have we done this in the past? What are the recollections, and the attitudes, of our graduates? People often say that the mathematics they learned in school and the mathematics that they use in their lives are very different and have little if anything to do with each other. Here's an example: The textbook or the teacher may have asked how long it takes to drive 20 miles at 40 miles per hour, and accepted the answer of 30 minutes. But how does all this come up in everyday life? When you live 20 miles from the airport, the speed limit is 40 mph, and your cousin is due at 6:00 pm, does that mean you leave at 5:30 pm? Your actual thinking may be quite different. This is rush hour. There are those intersections at which you don't have the right of way. How long will it take to find a place to park? If you take the back way, the average drive may take longer, but there is much less variability in the total drive time. And you'd better find out when the plane is expected. But don't forget that the arrival time the airline's website gives you is the time the plane is expected to touch down on the runway, not when it will start discharging passengers at the gate. And so forth. Contrasting these two thought processes, there is no wonder that graduates don't see the connection between mathematics and real life.

We said at the beginning that in mathematical modeling, both the real-world situation and the mathematics are "taken seriously". What does that mean? It means that the words and images from outside of mathematics are not just idle decorations. It means that the *size* of any numbers involved is realistic, that the *precision* of the numbers is realistic, that the question asked is one that you would ask in the real world. It means that you have considered what aspects of the real-world situation you intend to keep and what aspects you will ignore.

A mathematical model, as we have seen, begins with a situation in the real world which we wish to understand. The particular branch of mathematics that you will end up using may not be known when you start. But then how do you know when and where in the curriculum to discuss a certain modeling problem? If you put it in a section on plane geometry, then students will look for a plane geometric model! Is that what you want? An answer to this difficulty, which is quite real, is that, as in all of mathematics, the learning and the pedagogy are spiral and you return to a major idea many times. In the student's first experiences with modeling, the particular mathematics to be used will be quite obvious, and that's fine. Later on, the student may have to consider some alternatives ("Should I try plane geometry, or analytic geometry, or vectors?"), but may very well need help in finding what these alternatives might be, and how to think about the consequences of picking any particular one. At an advanced level, such hints will, we hope, become less necessary.

The Variety of Modules

We have seen that modeling arises in many major disciplines within science, engineering, and even social sciences. As such, it will be at the heart of courses in many disciplines, and at the heart of many varied careers. Mathematical modeling is also an important aspect of everyday life, where everyone will be better off if they become comfortable with it. It enters many facets of intelligent citizenship. Which kinds of situations do you want to emphasize in school? It is tempting to use modeling as an opportunity to get students thinking about the big issues of our time: world peace, health care, the economy, or the environment. The main point is to develop a favorable disposition and comfort with mathematical modeling, and big issues don't often fit into modules with two lessons.

So, tempting as it may be, the contents of this handbook do not attack any of the major problems of the world. There are *some* modules that can be considered as giving a foretaste of a whole discipline: *A Model Solar System* points towards Newton, Kepler, and the laws of motion and astronomy. Periodic phenomena also appear in both natural and man-made systems, as can be seen in *Sunrise, Sunset*. *A Bit of Information* gives a taste of the very beginning of information theory, and *State Apportionment* starts some thinking about that particular fair division problem. An introduction to the modeling of epidemics can be associated with *Viral Marketing*.

Both continuous and discrete mathematics are important for modeling. *Bending Steel* and *Water Down the Drain* are examples of continuous problems. An important aspect of a modeling disposition is the ability to make "back-of-the-envelope" estimates that give insight into phenomena that are sometimes surprising. Both *Bending Steel* and the extension to *Water Down the Drain* partake of this aspect of modeling. On the other hand, *A Tour of Jaffa* is discrete, and *Sunken Treasure* has discrete, continuous, and even experimental aspects. And it is sometimes surprising that functions with piecewise definitions occur in the real world as often as they do. Such a problem involves both continuous and discrete thinking. *For the Birds* gives an unexpected example.

Quite naturally, the modules involve a wide variety of high school mathematical topics. Looking for a function with particular properties is at the heart of *A Bit of Information* (logarithms) and of *Rating Systems* (a logistic curve). Another method of looking for a function is to fit a curve to data, which is part of *A Model Solar System*. It is also part of the mathematical modeling process to progress through various areas of mathematics as you become more adept at a particular modeling situation. Thus geometry, algebra, and trigonometry are all part of the development of *Narrow Corridor*. *Sunken Treasure*, besides using a variety of forms of mathematical reasoning, even suggests using physics in order to do mathematics!

A number of other important mathematical ideas arise in the course of this collection of modeling problems. For example, in connection with several modules involving probability and statistics, the notion of optimal stopping occurs more than once. It is the central idea in *Picking a Painting* and has an important role in *The Whe to Play*. The Intermediate Value Theorem has a crucial role in *Unstable Table*, a delightful everyday-life application of mathematics towards having a comfortable meal in a restaurant. The logistic curve shows up in *Rating Systems* and Voronoi diagrams in *Gauging Rainfall*. Simple everyday-life situations are found in *For the Birds*, *Estimating Temperatures*, and *Changing It Up*. We do have one whimsical problem, *Flipping for a Grade*.

A fundamental aspect of mathematical modeling, as is emphasized many times in the *Common Core State Standards for Mathematics*, in the literature on modeling, and in the present work, is the fact that every model downplays certain aspects of reality, which in turn means that the mathematical results eventually have to be checked against reality. This may lead to successive deepening of the models, which shows up particularly in *Narrow Corridor*, *A Tour of Jaffa*, and *Unstable Table*. This may be viewed as a new facet of Polya's dictum, "look for a simpler problem".

It is time to bring this introductory essay to a close. We propose two codas, one for mathematicians and one for mathematics educators.

Coda for Mathematicians

We have discussed a number of examples to show the variety of experiences which this collection is intended to encompass. They illustrate situations from everyday life, from citizenship, and from major quantitative disciplines, situations chosen because they lend themselves to brief introductory experiences in mathematical modeling. Don't get the impression that all of this is an unnatural demand on mathematics education. Far from it, it strengthens the affinity between pure mathematics and its applications. The heart of mathematical modeling, as we have seen, is problem formulating before problem solving. So often in mathematics, we say "prove the following theorem" or "solve the following problem". When we start at this point, we are ignoring the fact that finding the theorem or the right problem was a large part of the battle. By emphasizing the problem finding aspect, mathematical modeling brings back to mathematics education this problem finding aspect of our subject, and greatly reinforces the unity of the total mathematical experience.

Coda for Mathematics Educators

Probably 40 years ago, I was an invited guest at a national summer conference whose purpose was to grade the AP Examinations in Calculus. When I arrived, I found myself in the middle of a debate occasioned by the need to evaluate a particular student's solution of a problem. The problem was to find the volume of a particular solid which was inside a unit three-dimensional cube. The student had set up the relevant integrals correctly, but had made a computational error at the end and came up with an answer in the millions. (He multiplied instead of dividing by some power of 10.) The two sides of the debate had very different ideas about how to allocate the ten possible points. Side 1 argued, "He set everything up correctly, he knew what he was doing, he made a silly numerical error, let's *take off* a point." Side 2 argued, "He must have been sound asleep! How can a solid inside a unit cube have a volume in the millions?! It shows no judgment at all. Let's *give* him a point."

My recollection is that Side 1 won the argument, by a large margin. But now suppose the problem had been set in a mathematical modeling context. Then it would no longer be an argument just from the traditional mathematics point of view. In a mathematical modeling situation, pure mathematics loses some of its sovereignty. The quality of a result is judged not only by the correctness of the mathematics done within the idealized mathematical situation, but also by the success of the confrontation with reality at the end. If the result doesn't make sense in terms of the original situation in the real world, it's not an acceptable solution.

How would you vote?