HOMOTOPY THEORY AND ARITHMETIC GEOMETRY I: ÉTALE HOMOTOPY TYPE AND GROTHENDIECK'S ANABELIAN CONJECTURES

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Introduction

One might be interested in solutions to polynomial equations with $\mathbb{Q}$ or $\mathbb{Z}$ coefficients. This leads to the notion of a *scheme*. A scheme X is locally associated to a collection of polynomial equations and it records to solutions $X(R)$ in any ring R containing the coefficients. Note $X(\mathbb{C})$ has a topology coming from $\mathbb{C}$.

Example. Consider $y^2 = f(x) = (x - r_1)(x - r_2) \cdots (x - r_{2n})$ where the coefficients $f(x) \in \mathbb{Q}[x]$. Let X be the associated scheme. We have a map $X(\mathbb{C}) \to \mathbb{C}$ defined by $(x, y) \mapsto x$. Sitting above any point except for the r_i we have two different points in $X(\mathbb{C})$. Let's suppose the r_i are distinct. This defines a 2-sheeted covering space. If we cut intervals from r_1 to r_2, …, r_{2n-1} to r_{2n} and similarly in the covering spaces, we can glue the two sheets of the covering space together after flipping the top copy. Up to homotopy, we're gluing in cylinders connecting the slits in the two sheets of the cover. This gives us a surface of genus $n - 1$ minus two points (the points at infinity).

It's a lovely fact that the topology of $X(\mathbb{C})$ gives information about $X(\mathbb{Q})$, which are $X(\mathbb{Z})$ points for X a subset of projective space.

Theorem (Faltings)**.** *For genus* ≥ 2, $X(\mathbb{Q}) < \infty$.

So something simple like the genus gives us solutions over much more delicate fields.

On a much more immediate level $X(\mathbb{R}) = X(\mathbb{C})^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}$; i.e., the $\mathbb{R}$-points are the fixed points of the conjugation action on the $\mathbb{C}$-points. We would like to have an action of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ on $X(\mathbb{C})$, but this does not exist. So instead, we'll replace $X(\mathbb{C})$ by the Étale homotopy type.

Étale homotopy type. This was introduced by Artin-Mazur and further refined by Friedlander. First we'll justify the desire to replace the $\mathbb{C}$-points by the Étale homotopy type. For schemes over $\mathbb{C}$ (only considering rings that are extensions of $\mathbb{C}$), The Étale homotopy type $\text{Ét}(X)$ is a profinite completion of $X(\mathbb{C})$.

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Let $\mathcal{S}$ denote spaces. Then $\mathbf{pro}\mathcal{S}$ is the category of prospaces defined by formally adding inverse limits to the category of spaces. We write $\{Z_\alpha\}_{\alpha\in A}$ which we think of as being the formal limit of an inverse system $\lim_A Z$.

Let C be the class of finite groups. Let $C\mathcal{S}$ be the class of spaces whose homotopy groups lie in C. We have an inclusion functor $\mathbf{pro}C\mathcal{S} \hookrightarrow \mathbf{pro}\mathcal{S}$ has a left adjoint and this is the *profinite completion*. We can regard a space as a prospace (by just taking the constant inverse system). This defines

$$\mathcal{S} \to \mathbf{pro}\mathcal{S} \to \mathbf{pro}C\mathcal{S}$$

which is how we usually think of the profinite completion functor.

Under certain hypotheses the Étale points are the profinite completion of the $\mathbb{C}$-points.

Theorem (Artin-Mazur [AM, 12.9, 11.1])**.** *For X of finite type over $\mathbb{C}$, locally Noetherian, geometrically unibranched, then there is a canonical $X(\mathbb{C}) \to \acute{E}t(X)$ which is profinite completion.*

Profinite completion gets denoted by $(-)^\wedge$.

Example. $K(G,1)^\wedge = K(\hat{G},1)$ where $G \to \hat{G}$ is the profinite completion of G, i.e., the initial map from G to an inverse limit of finite groups.

Construction motivating the definition of the Étale homotopy type. Let X be a topological space and $\mathcal{U}$ an open cover of X. We have $\coprod_{U\in\mathcal{U}} U \to X$. We'll define a category whose space of objects ins $\coprod_{U\in\mathcal{U}} U$ and whose space of morphisms is the fiber product $(\coprod_{U\in\mathcal{U}} U) \times_X (\coprod_{U\in\mathcal{U}} U)$. The *nerve* of $\mathcal{U}$ is the simplicial space with $N(\mathcal{U})_n$ the n-fold fiber product of $\coprod_{U\in\mathcal{U}} U$ over X. Then you get

$$|N(\mathcal{U})_\bullet| \to X$$

that is a weak equivalence so long as X has a partition of unity.

We likewise have a map

$$\begin{tikzcd} |N(\mathcal{U})_\bullet| \arrow[r] \arrow[d] & X \\ |\pi_0 N(\mathcal{U})| & \end{tikzcd}$$

The vertical is a weak equivalence when the intersections are weakly equivalent to a disjoint union of points.

Example. Let $X = S^1$ and let $\mathcal{U}$ be the open cover consisting of two intervals U_1 and U_2. The 0-simplices of $N(\mathcal{U})$ are $U_1 \coprod U_2$ while the 1-simplices are $U_1 \cap U_2$. So the geometric realization looks like two arcs (representing U_1 and U_2) glued to two rectangles $(\Delta^1 \times (U_1 \cap U_2))$, which is indeed weakly equivalent to S^1.

We have $X \simeq \{|\pi_0 N(\mathcal{U})|\}_{\mathcal{U}\text{ all covers}}$ is in $\mathbf{pro}(\mathcal{S})$. This is how we'll associate a topological space to a scheme. (From now on we'll associate a simplicial space to its geometric realization and drop the $|-|$.)

Further motivation. We can generalize the notion of a covering by opens to a Grothendieck topology. Covers will still look like $\coprod_{U\in\mathcal{U}} U \to X$ but the map $U \to X$ isn't required to be the inclusion of an open set. This leads to an *Étale topology* that identifies certain maps $U \to X$ as "covers." These are composites $U \to V \to X$ where $V \to X$ is the inclusion of an open subset and $U \to V$ is a covering space. The point of this is that such things are still schemes (cut out by polynomial equations). This will allow us to consider Vs whose intersections aren't disjoint unions of contractible spaces but are instead $K(\pi,1)$s.

Example. $X = K(G,1)$. Let $\mathcal{U} = \{\tilde{X} \to X\}$ be the universal cover. Then $N(\mathcal{U})$ has n-simplices the n-fold fiber product. The 2-fold fiber product consists of pairs of points with common image and a path between them (a deck transformation, i.e., a group element). So $N(\mathcal{U}) = \tilde{X} \times NG$, where NG is the nerve of G regarded as a one object category. So $\pi_0(N\mathcal{U}) = K(G,1)$.

Theorem (Sullivan [S])**.** *More generally, let $\mathcal{V}$ be a covering by open sets of a topological space X such that all intersections of elements of $\mathcal{V}$ are $K(G,1)$s. Let $\mathcal{U} = \{U \to V \to X\}_{V \in \mathcal{V}}$ where the first map is the universal cover. Then $\pi_0 N(\mathcal{U}) \simeq X$.*

Example (loc cit)**.** The Hopf map. $X = \mathbb{C}^2 - \{0\} \simeq S^3$. Let $\mathcal{V}$ include $V_1 = \{(x,y) \in X \mid x \neq 0\} \simeq S^1$, $V_2 = \{(x,y) \in X \mid y \neq 0\} \simeq S^1$, and $V_1 \cap V_2 \simeq S^1 \times S^1$. Then

$$X = \operatorname{hocolim}(S^1 \leftarrow S^1 \times S^1 \to S^1).$$

Let $\mathcal{U} = \{\tilde{V_1} \to X, \tilde{V_2} \to X, V_1 \tilde{\cap} V_2 \to X\}$. Then

$$\pi_0 N(\mathcal{U}) = \operatorname{hocolim}(N(\mathbb{Z}) \leftarrow N(\mathbb{Z} \oplus \mathbb{Z}) \to N(\mathbb{Z})) \simeq S^3.$$

Similarly,

$$S^2 = \operatorname{hocolim}(* \leftarrow N(\mathbb{Z}) \to *)$$

as the suspension of S^1. The Hopf map is the map of hocolims induced by the map between diagrams that sends $m \oplus n$ to $m - n$.

The definition of the Étale homotopy type

The category of schemes has the Zariski topology and also the étale topology, which includes finite covering spaces of Zariski opens. There is a notion of π_0. So for any $\mathcal{U}$, an étale cover of X, we still have the nerve of $\mathcal{U}$ with $N(\mathcal{U})_n$ the n-fold fiber product of $\coprod_{U \in \mathcal{U}} U$ over X.

Definition. If X is quasi-projective, $\text{Ét}(X)$ is the prosimplicial set $\{\pi_0 N(\mathcal{U})\}_{\mathcal{U}}$ where we're indexed by the étale covers $\mathcal{U}$ of X.

For schemes over $\mathbb{C}$ this is the profinite competition of the analytic topology of the $\mathbb{C}$-points.

There are some caveats:

- For general X, the $N(\mathcal{U})$ are replaced by hypercovers.
- The category of étale covers of X is only cofiltered (left-filtering) up to homotopy.

The plan is to use the Étale homotopy type to discuss solutions to polynomial equations taking Grothendieck's anabelian conjectures as a method. In an ideal world, we'd give some other points of view about the Étale homotopy type before stating the anabelian conjectures with a motivating example. There are some sacrifices that have to be made. Next time we'll state the anabelian conjectures and connect them up with $\mathbb{A}^1$-homotopy theory.

Other points of view. We can view $\{N(\mathcal{U})\}_{\mathcal{U} \text{ cover } X}$ as a cofibrant replacement of X. So then we can think of $\text{Ét}(X)$ as $\mathbb{L}\pi_0$, the left derived functor of π_0.

Alternatively, we can think of sheaves on the étale site of X. There is the global sections functor $\Gamma_* \colon \mathbf{Sh}(\text{étale site of } X) \to \mathcal{S}$. The constant sheaf functor Γ^* is left adjoint to this.

After passing to procategories, π_0 is left adjoint to Γ^*. Let $\{Z_\alpha\}_{\alpha\in A} \in \mathbf{pro}\mathcal{S}$. Then

$$\begin{aligned}
\mathrm{Mor}(\acute{\mathrm{E}}\mathrm{t}(X), \{Z_\alpha\}_{\alpha\in A}) &= \mathrm{Mor}(\mathbb{L}\pi_0 X, \{Z_\alpha\}_{\alpha\in A}) \\
&= \mathrm{Mor}(X, \Gamma^*\{Z_\alpha\}_{\alpha\in A}) \\
&= \lim_{\alpha\in A} \mathrm{Mor}(X, \Gamma^* Z_\alpha) \\
&= \lim \Gamma_*\Gamma^* Z_\alpha.
\end{aligned}$$

The endofunctor $\Gamma_*\Gamma^* \colon \mathcal{S} \to \mathcal{S}$ that we can regard as a prospace. Lurie *Higher topos theory* says this is the shape of the étale topos, i.e., is $\acute{\mathrm{E}}\mathrm{t}(X)$.

References

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[S] Sullivan. Geometric Topology: Localization, Periodicity, and Galois Symmetry. The MIT notes.

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