LIMITS OF QUASI-CATEGORIES WITH (CO)LIMITS

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Introduction

This talk concerns quasi-categories which are a model for $(\infty, 1)$-categories, which are categories with objects, 1-morphisms, 2-morphisms, 3-morphisms, and so on, with everything above level one invertible. Specifically, a quasi-category is a simplicial set in which any inner horn has a filler. We think of the horn filling as providing a weak composition law for morphisms in all dimensions.

Our project is to redevelop the foundational category theory of quasi-categories (previously established by Joyal, Lurie, and others) in a way that makes it easier to learn. In particular, the proofs more closely resemble classical categorical proofs. Today I want to illustrate this by mentioning one new theorem (to appear on the arXiv on Monday) and then describing as much as I can about its proof.

Theorem. *Homotopy limits of quasi-categories that have and functors that preserve X-shaped (co)limits have X-shaped (co)limits, and the legs of the limit cone preserve them.*

Here X can be any simplicial set. X-shaped colimits might be pushouts, filtered colimits, initial objects, colimits of countable sequences, and so on. The two theorems (for X-shaped limits or colimits) are dual, so I won't mention colimits further.

By "homotopy limits" I mean Bousfield-Kan style homotopy limits, which are defined via a particular formula. Here there is no dual result for homotopy colimits. This has to do with the fact that the quasi-categories are the fibrant objects in a model structure on simplicial sets. And actually, the result that we prove is for a more general class of limits, including the homotopy limits, that I will describe along the way.

Today I'll focus on a special case of the theorem: quasi-categories admitting and functors preserving $\emptyset$-shaped limits, aka terminal objects. In fact, the general case reduces to this special one, though I won't have time to explain how.

Warmup

To warm up, let's prove the following result:

Theorem. *The homotopy limit of a diagram of quasi-categories is a quasi-category.*

Here a diagram means a simplicial functor $D\colon \mathbf{A} \to \mathbf{qCat}_\infty$. Here $\mathbf{qCat}_\infty$ is the simplicially enriched category of quasi-categories, defined to be a full subcategory of simplicial sets. The domain $\mathbf{A}$ is either a small category or a small simplicial category; we care about both cases.

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Projective cofibrant weighted limits. Homotopy limits are examples of *projective cofibrant weighted limits*. By a weight, in the context of the diagram D above, I mean a simplicial functor $W\colon \mathbf{A} \to \mathbf{sSet}$. For instance:

Example. Taking the weight to be $N(\mathbf{A}/-)\colon \mathbf{A} \to \mathbf{sSet}$, the corresponding limit notion is the *Bousfield-Kan homotopy limit*.

Example. If $\mathbf{A}$ is the category $\bullet \to \bullet \leftarrow \bullet$, we might define W to be the functor with image $\Delta^0 \xrightarrow{d^1} \Delta^1 \xleftarrow{d^0} \Delta^0$. The weighted limit in then a *comma object*.

Example. There is a weight whose weighted limit defines the quasi-category of *homotopy coherent algebras* for a homotopy coherent monad. Some of you heard me talk about this last week at the Joint Meetings.

A weight W is *projective cofibrant* if $\emptyset \to W$ is a retract of a composite of pushouts of coproducts of maps $\partial\Delta^n \times \mathbf{A}(a,-) \to \Delta^n \times \mathbf{A}(a,-)$ for $n \geq 0$ and $a \in \mathbf{A}$. These are exactly the cofibrant objects in the projective model structure on the category of simplicial functors $\mathbf{sSet}^{\mathbf{A}}$.

The weighted limit is a bifunctor

$$\text{weighted limit}\colon (\text{weight})^{\text{op}} \times \text{diagram} \xrightarrow{\{-,-\}} \text{limit object}$$

that is completely characterized by the following two axioms:

(i) $\{\mathbf{A}(a,-), D\} = Da$, i.e., the weighted limit weighted by a representable functor just evaluates the diagram at that object.
(ii) $\{-, D\}$ sends colimits in the weight to limits in the weighted limit.

Proof strategy. These two facts combine to give us a strategy for the proof of our warm-up theorem, which I will now restate:

Theorem. *A projective cofibrant weighted limit of a diagram of quasi-categories is a quasi-category.*

Proof. It suffices to show that $\mathbf{qCat}_\infty$ is closed under

(i) splittings of idempotents (i.e., retracts)
(ii) limits of towers of isofibrations
(iii) pullbacks of isofibrations
(iv) products
(v) cotensors $(-)^Y$ with any simplicial set Y

and moreover that a monomorphism $X \hookrightarrow Y$ induces an isofibration $(-)^Y \to (-)^X$ of quasi-categories. Here an *isofibration* is a fibration between fibrant objects in the Joyal model structure on simplicial sets. All of the facts (i)-(iii) follow immediately from the fact that the quasi-categories are the fibrant objects in this monoidal model structure. $\square$

Quasi-categories with terminal objects

Now let us consider $\mathbf{qCat}_{\emptyset,\infty} \subset \mathbf{qCat}_\infty$, the simplicial category of quasi-categories admitting and functors preserving terminal objects (and all higher morphisms whose vertices are functors preserving terminal objects). Our aim is to prove:

Theorem. *A projective cofibrant weighted limit of a diagram in* $\mathbf{qCat}_{\emptyset,\infty}$ *is in* $\mathbf{qCat}_{\emptyset,\infty}$.

As before, it suffices to show that $\mathbf{qCat}_{\emptyset,\infty}$ is closed under the classes of limits (i)-(v) and that cotensors with monomorphisms induce isofibrations that preserve terminal objects. Before going any further, we should define a terminal object in the quasi-categorical context.

Definition. A vertex t in a quasi-category A is *terminal* if any of the following equivalent conditions are satisfied:

(i) Any sphere in A whose final vertex is t has a filler.

$$\begin{tikzcd} \Delta^0 \arrow[r, "{[n]}"'] \arrow[rr, bend left, "t"] & \partial\Delta^n \arrow[r] \arrow[d] & A \\ & \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

(ii) There is an adjunction of quasi-categories $\begin{tikzcd} A \arrow[r, bend left, "!"] \arrow[r, phantom, "\perp"] & \Delta^0 \arrow[l, bend left, "t"] \end{tikzcd}$

(iii) For all simplicial sets X, the constant functor $X \xrightarrow{!} \Delta^0 \xrightarrow{t} A$ is terminal in $h(A^X)$, the homotopy category of the mapping space A^X.

Definitions (ii) and (iii) refer implicitly to $\mathbf{qCat}_2$, the strict 2-category of quasi-categories, defined by applying the homotopy category functor to the hom-spaces of $\mathbf{qCat}_\infty$.

To conclude, I'll quickly prove parts (iv), (i), and (v) of the theorem. Parts (ii) and (iii) are no more difficult, but require some basic facts about isofibrations and terminal objects.

Lemma (products)**.** *Suppose $t_i \in A_i$ is terminal. Then $(t_i)_{i\in I} \in \prod_{i\in I} A_i$ is terminal.*

Proof. Given a sphere

$$\begin{tikzcd} \Delta^0 \arrow[r, "{[n]}"'] \arrow[rr, bend left, "(t_i)_{i\in I}"] \arrow[rrr, bend left=40, "t_i"] & \partial\Delta^n \arrow[r] \arrow[d] & \prod_{i\in I} A_i \arrow[r, "\pi_i"] & A_i \\ & \Delta^n \arrow[ur, dashed] \arrow[urr, dashed] & & \end{tikzcd}$$

the fact that the $t_i \in A_i$ are terminal for each i defines the components of the filler. Note that each projection π_i prefers this particular terminal object. This implies that it preserves all terminal objects because all terminal objects are isomorphic. □

Lemma (idempotents)**.** *Suppose $t \in A$ is terminal, $e\colon A \to A$ is an idempotent ($e^2 = e$), and e preserves terminal objects (so et $\in$ A is terminal). We split the idempotent by forming the equalizer*

$$A^e \rightarrowtail \mathrm{eq}(A \overset{e}{\underset{\mathrm{id}}{\rightrightarrows}} A)$$

Then et $\in A^e$ is terminal.

Proof. Observe that $e^2 = e$ implies that $et \in A^e$. Given a sphere

$$\begin{tikzcd} \Delta^0 \arrow[r, "{[n]}"'] \arrow[rr, bend left, "et"] & \partial\Delta^n \arrow[r] \arrow[d] & A^e \arrow[d] \\ & \Delta^n \arrow[r, dashed, "a"'] \arrow[ur, dashed, "ea"'] & A \end{tikzcd}$$

the fact that et is terminal in A implies there exists a filler $a\colon \Delta^n \to A$ for the composite sphere in A. One can check that $ea\colon \Delta^n \to A^e$ fills the sphere in A^e $\square$

Products and idempotents are both conical limits. For cotensors, we'll switch to the equivalent definition (iii).

Lemma (cotensors)**.** *Suppose $t \in A$ is terminal and Y is a simplicial set. Then $Y \xrightarrow{!} \Delta^0 \xrightarrow{t} A$ is terminal in A^Y.*

Proof. To say $t \in A$ is terminal is to say that for any simplicial set X and any map $X \to A$ there is a unique 2-cell

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "!"'] & \arrow[d, phantom, "\Downarrow\exists!"] & A \\ & \Delta^0 \arrow[ur, "t"'] & \end{tikzcd}$$

By 2-cell I mean a morphism in $h(A^X)$, i.e., an endpoint-preserving homotopy class of 1-simplices in A^X. This is true for any X so in particular, we have a unique 2-cell as on the left below.

$$\begin{tikzcd} X \times Y \arrow[rr] \arrow[dr, "!"'] & \arrow[d, phantom, "\Downarrow\exists!"] & A \\ & \Delta^0 \arrow[ur, "t"'] & \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} X \arrow[rr] \arrow[dr, "!"'] & \arrow[d, phantom, "\Downarrow\exists!"] & A^Y \\ & (\Delta^0)^Y \arrow[ur, "t^Y"'] & \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[rr] \arrow[dr, "!"'] & \arrow[d, phantom, "\Downarrow\exists!"] & A^Y \\ & \Delta^0 \arrow[ur, "t\cdot !"'] & \end{tikzcd}$$

The 2-category of quasi-categories is cartesian closed, so applying the 2-adjunction $- \times Y \dashv (-)^Y$, this transposes to a unique 2-cell in the triangle on the right. By (iii) this says exactly that the constant map at t is terminal in A^Y. $\square$

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