LIMITS OF QUASI-CATEGORIES WITH (CO)LIMITS

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INTRODUCTION

This talk concerns quasi-categories which are a model for $(\infty, 1)$ -categories, which are categories with objects, 1-morphisms, 2-morphisms, 3-morphisms, and so on, with everything above level one invertible. Specifically, a quasi-category is a simplicial set in which any inner horn has a filler. We think of the horn filling as providing a weak composition law for morphisms in all dimensions.

Our project is to redevelop the foundational category theory of quasi-categories (previously established by Joyal, Lurie, and others) in a way that makes it easier to learn. In particular, the proofs more closely resemble classical categorical proofs. Today I want to illustrate this by mentioning one new theorem (to appear on the arXiv on Monday) and then describing as much as I can about its proof.

Theorem. Homotopy limits of quasi-categories that have and functors that preserve X-shaped (co)limits have X-shaped (co)limits, and the legs of the limit cone preserve them.

Here X can be any simplicial set. X-shaped colimits might be pushouts, filtered colimits, initial objects, colimits of countable sequences, and so on. The two theorems (for X-shaped limits or colimits) are dual, so I won't mention colimits further.

By "homotopy limits" I mean Bousfield-Kan style homotopy limits, which are defined via a particular formula. Here there is no dual result for homotopy colimits. This has to do with the fact that the quasi-categories are the fibrant objects in a model structure on simplicial sets. And actually, the result that we prove is for a more general class of limits, including the homotopy limits, that I will describe along the way.

Today I'll focus on a special case of the theorem: quasi-categories admitting and functors preserving \emptyset -shaped limits, aka terminal objects. In fact, the general case reduces to this special one, though I won't have time to explain how.

WARMUP

To warm up, let's prove the following result:

Theorem. The homotopy limit of a diagram of quasi-categories is a quasi-category.

Here a diagram means a simplicial functor $D: \mathbf{A} \to \mathbf{qCat}_{\infty}$. Here \mathbf{qCat}_{∞} is the simplicially enriched category of quasi-categories, defined to be a full subcategory of simplicial sets. The domain \mathbf{A} is either a small category or a small simplicial category; we care about both cases.

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Projective cofibrant weighted limits. Homotopy limits are examples of *projective cofibrant weighted limits*. By a weight, in the context of the diagram D above, I mean a simplicial functor $W: \mathbf{A} \rightarrow \mathbf{sSet}$. For instance:

Example. Taking the weight to be $N(\mathbf{A}/-)$: $\mathbf{A} \rightarrow \mathbf{sSet}$, the corresponding limit notion is the *Bousfield-Kan homotopy limit*.

Example. If **A** is the category $\bullet \to \bullet \leftarrow \bullet$, we might define *W* to be the functor with image $\Delta^0 \xrightarrow{d^1} \Delta^1 \xleftarrow{d^0} \Delta^0$. The weighted limit in then a *comma object*.

Example. There is a weight whose weighted limit defines the quasi-category of *homotopy coherent algebras* for a homotopy coherent monad. Some of you heard me talk about this last week at the Joint Meetings.

A weight *W* is *projective cofibrant* if $\emptyset \to W$ is a retract of a composite of pushouts of coproducts of maps $\partial \Delta^n \times \mathbf{A}(a, -) \to \Delta^n \times \mathbf{A}(a, -)$ for $n \ge 0$ and $a \in \mathbf{A}$. These are exactly the cofibrant objects in the projective model structure on the category of simplicial functors **sSet**^A.

The weighted limit is a bifunctor

weighted limit: (weight)^{op} × diagram $\xrightarrow{\{-,-\}}$ limit object

that is completely characterized by the following two axioms:

- (i) $\{A(a, -), D\} = Da$, i.e., the weighted limit weighted by a representable functor just evaluates the diagram at that object.
- (ii) $\{-, D\}$ sends colimits in the weight to limits in the weighted limit.

Proof strategy. These two facts combine to give us a strategy for the proof of our warm-up theorem, which I will now restate:

Theorem. A projective cofibrant weighted limit of a diagram of quasi-categories is a quasi-category.

Proof. It suffices to show that \mathbf{qCat}_{∞} is closed under

- (i) splittings of idempotents (i.e., retracts)
- (ii) limits of towers of isofibrations
- (iii) pullbacks of isofibrations
- (iv) products
- (v) cotensors $(-)^Y$ with any simplicial set Y

and moreover that a monomorphism $X \hookrightarrow Y$ induces an isofibration $(-)^Y \to (-)^X$ of quasicategories. Here an *isofibration* is a fibration between fibrant objects in the Joyal model structure on simplicial sets. All of the facts (i)-(iii) follow immediately from the fact that the quasi-categories are the fibrant objects in this monoidal model structure.

QUASI-CATEGORIES WITH TERMINAL OBJECTS

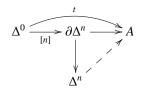
Now let us consider $\mathbf{qCat}_{\emptyset,\infty} \subset \mathbf{qCat}_{\infty}$, the simplicial category of quasi-categories admitting and functors preserving terminal objects (and all higher morphisms whose vertices are functors preserving terminal objects). Our aim is to prove:

Theorem. A projective cofibrant weighted limit of a diagram in $\mathbf{qCat}_{0,\infty}$ is in $\mathbf{qCat}_{0,\infty}$.

As before, it suffices to show that $\mathbf{qCat}_{\emptyset,\infty}$ is closed under the classes of limits (i)-(v) and that cotensors with monomorphisms induce isofibrations that preserve terminal objects. Before going any further, we should define a terminal object in the quasi-categorical context.

Definition. A vertex *t* in a quasi-category *A* is *terminal* if any of the following equivalent conditions are satisfied:

(i) Any sphere in A whose final vertex is t has a filler.



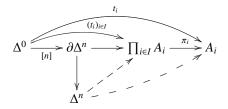
- (ii) There is an adjunction of quasi-categories $A \underbrace{\perp}^{!} \Delta^{0}$
- (iii) For all simplicial sets X, the constant functor $X \xrightarrow{!} \Delta^0 \xrightarrow{t} A$ is terminal in $h(A^X)$, the homotopy category of the mapping space A^X .

Definitions (ii) and (iii) refer implicitly to \mathbf{qCat}_2 , the strict 2-category of quasi-categories, defined by applying the homotopy category functor to the hom-spaces of \mathbf{qCat}_{∞} .

To conclude, I'll quickly prove parts (iv), (i), and (v) of the theorem. Parts (ii) and (iii) are no more difficult, but require some basic facts about isofibrations and terminal objects.

Lemma (products). Suppose $t_i \in A_i$ is terminal. Then $(t_i)_{i \in I} \in \prod_{i \in I} A_i$ is terminal.

Proof. Given a sphere



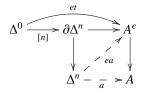
the fact that the $t_i \in A_i$ are terminal for each *i* defines the components of the filler. Note that each projection π_i prefers this particular terminal object. This implies that it preserves all terminal objects because all terminal objects are isomorphic.

Lemma (idempotents). Suppose $t \in A$ is terminal, $e: A \to A$ is an idempotent ($e^2 = e$), and e preserves terminal objects (so $et \in A$ is terminal). We split the idempotent by forming the equalizer

$$A^e \rightarrow \operatorname{eq}(A \xrightarrow[id]{e} A)$$

Then $et \in A^e$ *is terminal.*

Proof. Observe that $e^2 = e$ implies that $et \in A^e$. Given a sphere

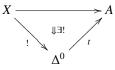


the fact that *et* is terminal in A implies there exists a filler $a: \Delta^n \to A$ for the composite sphere in A. One can check that $ea: \Delta^n \to A^e$ fills the sphere in A^e

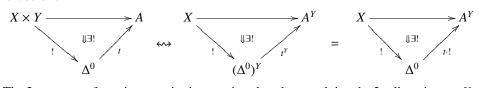
Products and idempotents are both conical limits. For cotensors, we'll switch to the equivalent definition (iii).

Lemma (cotensors). Suppose $t \in A$ is terminal and Y is a simplicial set. Then $Y \xrightarrow{!} \Delta^0 \xrightarrow{t} A$ is terminal in A^Y .

Proof. To say $t \in A$ is terminal is to say that for any simplicial set X and any map $X \to A$ there is a unique 2-cell



By 2-cell I mean a morphism in $h(A^X)$, i.e., an endpoint-preserving homotopy class of 1-simplices in A^X . This is true for any X so in particular, we have a unique 2-cell as on the left below.



The 2-category of quasi-categories is cartesian closed, so applying the 2-adjunction $-\times Y + (-)^Y$, this transposes to a unique 2-cell in the triangle on the right. By (iii) this says exactly that the constant map at *t* is terminal in A^Y .

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