HOMOLOGICAL STABILITY FOR FAMILIES OF GROUPS I

NATHALIE WAHL

Introduction

I'm going to talk about homological stability, so I'll start by saying what I mean by this. I'll talk about stability for families of groups:

$$G_1 \hookrightarrow G_2 \hookrightarrow G_3 \hookrightarrow \cdots .$$

The sort of examples I have in mind are

- $G_n = \Sigma_n$, the symmetric group on n-letters. The maps $\Sigma_n \hookrightarrow \Sigma_{n+1}$ add an element and extend the permutations by examples.
- $G_n = GL_n(\mathbb{Z})$. Here $GL_n(\mathbb{Z}) \hookrightarrow GL_{n+1}(\mathbb{Z})$ is the map

$$(A) \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

Write $H_i(G_n) = H_i(G_n, \mathbb{Z}) = H_i(BG, \mathbb{Z})$ for *group homology*. By *homological stability* I mean $H_i(G_n) \xrightarrow{\cong} H_i(G_{n+1})$ for $i \ll n$ (the stable range growing with n).

Remark. There is the "stable group" $G_\infty = \cup_{n\geq 1} G_n$. $\{G_n\}$ satisfies homological stability if and only if $H_i(G_n) \cong H_i(G_\infty)$ for $i \ll n$. Then $H_*(G_\infty)$ is the stable homology.

Q. Some questions:

(i) Does stability ever happen?
(ii) When should I expect it happens?
(iii) What's a criterion?

Known examples

Example (Nakoka)**.** Symmetric groups: $H_i(\Sigma_n) \to H_i(\Sigma_{n+1})$ is an isomorphism if $i \leq \frac{n}{2}$.

Example (Arnol'd)**.** Braid groups: define $B_n = \pi_1 \mathrm{Conf}(n, \mathbb{R}^2)$. We have $H_i(B_n) \xrightarrow{\cong} H_i(B_{n+1})$ if $i \leq \frac{n}{2}$.

Example (Quillen)**.** General linear groups: $H_i(GL_n(R)) \to H_i(GL_{n+1}(R))$ is an isomorphism $i \leq n-1$ when R is a field $\neq \mathbb{F}_2$. This is the theorem Quillen proved but it also holds for most rings, though I have to change the range. This is due to Charney, Vogtmann, Wagoner, ...

Example. Mapping class groups: Let M, N be manifolds of dimension $d = 2, 3$ (or other dimensions in principle). If $\dim M = \dim N$, you can define $M\#N = (M\backslash D^d) \cup_{S^{d-1}} (N \backslash D^d)$. We now have a family of manifolds M, $M\#N$, $M\#N\#N$, ... Fix M with non-empty manifold and you get

$$\pi_0 \mathrm{Diff}(M, \partial M) \hookrightarrow \pi_0 \mathrm{Diff}(M\#N, \partial M) \hookrightarrow \cdots .$$

Date: Connections for Women: Algebraic Topology — MSRI — 23 January, 2014.

It's not obvious that there is a map (this uses the boundary of M) and it's less obvious that the map is injective (but at least this is true in dimensions 2 and 3).

Theorem (Harer, Ivanov, Boldsen, Randall-Williams, Hatcher-W, W)**.** *The above map* $H_i(\pi_0\mathrm{Diff}(M\#_n N, \partial M)) \to H_i(\pi_0\mathrm{Diff}(M\#_{n+1} N, \partial M))$ *is an iso for* $i \leq \frac{n-2}{2}$ *for* $d = 2, 3$ *and* M, N *orientable (or* $\frac{n-3}{3}$ *if* $n = 2$ *and* N *is non-orientable).*

Example (Hatcher-Vogtmann)**.** Automorphisms of free groups: $F_n = \mathbb{Z} * \cdots * \mathbb{Z}$ the free group on n elements. Then $H_i(\mathrm{Aut}(F_n)) \xrightarrow{\cong} H_i(\mathrm{Aut}(F_{n+1}))$ for $i \leq \frac{n-2}{2}$.

A common framework

This is an extension of the set-up of Djanent-Vespa.

Idea. The groups in such families are all automorphism groups of objects in a very special type of category, which encodes "everything" about stability.

Suppose $(C, \oplus, 0)$ is a monoidal category where the unit 0 is initial.

Example. $C = \mathbf{FI}$, finite sets and injections. $\oplus = \coprod$ and $0 = \emptyset$.

Want such C satisfying:

(H1) $\mathrm{Aut}_C(B)$ acts transitively on $\mathrm{Hom}_C(A, B)$ for every $A, B \in C$.

Example. In **FI**, (H1) is satisfied (post-composition by automorphisms is transitive, though not free).

(H2) $\mathrm{Aut}_C(A) \to \mathrm{Aut}_C(A \oplus B)$ by $f \mapsto f \oplus \mathrm{id}_B$ is injective with image $\mathrm{Fix}(B)$, the maps $\phi \in \mathrm{Aut}_C(A \oplus B)$ so that

$$\begin{array}{ccc} B \cong 0 \oplus B & \xrightarrow{!\oplus \mathrm{id}_B} & A \oplus B \\ & \searrow^{!\oplus \mathrm{id}_B} & \downarrow \phi \\ & & A \oplus B \end{array}$$

Example. In **FI**, automorphisms of A map injectively to automorphisms of $A \coprod B$ and these are the only automorphisms of $A \coprod B$ that fix $B \hookrightarrow A \coprod B$.

The last axiom is the "connectivity" axiom. Pick $A, X \in C$. For $n \geq 1$, define a simplicial complex $S_n = S_n(A, X)$. The vertices are $\mathrm{Hom}_C(X, A \oplus X^{\oplus n})$. A collection $\{f_0, \ldots, f_p\}$ of $p+1$ distinct vertices forms a simplex if there exists $X^{\oplus p+1} \xrightarrow{f} A \oplus X^{\oplus n}$ so that the restriction along the j-th inclusion $X \to X^{\oplus p+1}$ is f_j for all $1 \leq j \leq p+1$.

If $\oplus$ is the coproduct, then any collection of vertices would form a simplex, but in general this is not the case. We think of this as some lifting condition.

Example. In **FI**, pick $A = \emptyset$, $X = \{*\}$. Then $S_n = S_n(\emptyset, \{*\})$ has vertices $\mathrm{Hom}(*, \coprod_n *) = \{1, \ldots, n\}$. The vertices $[i_0, \ldots, i_p]$ form a p-simplex if there exists a map $p+1 \to n$ with these components. This is always the case (if the vertices are distinct, as they must be). So any collection gives a simplex and our simplicial complex S_n is Δ^{n-1}.

(H3) $\tilde{H}_i(S_n(A, X)) = 0$ for $i \ll n$ (some range increasing with n).

Theorem. *For* $(C, \oplus, 0)$ *monoidal with 0 initial satisfying (H1)-(H3) (for some* A, X*) and* C *symmetric or "asymmetric" then the groups* $G_n = \mathrm{Aut}_C(A \oplus X^{\oplus n})$ *satisfy homological stability.*

If (H3) is satisfied with the range $i \leq \frac{n-2}{2}$, then $H_i(\mathrm{Aut}(A \oplus X^{\oplus n})) \xrightarrow{\cong} H_i(\mathrm{Aut}(A \oplus X^{\oplus n+1}))$ for $i \leq \frac{n-2}{2}$.

Example. For $C = \mathbf{FI}$, $A = \emptyset$, $X = \{*\}$, $G_n = \mathrm{Aut}(A \oplus X^{\oplus n}) = \Sigma_n$ the theorem recovers the stability for the symmetric groups. (The simplices are contractible so there is no connectivity problem.)

I'll show tomorrow that the other examples mentioned at the beginning also fit in this framework.

Proof. (Classical, due to Quillen.) Given $G_1 \hookrightarrow G_2 \hookrightarrow \cdots \hookrightarrow G_n \hookrightarrow \cdots$ find highly connected space S_n for all n with G_n acting on S_n with good transitivity and stabilizer properties. It's best if S_n is a simplicial complex or something filtered. Then there is a spectral sequence that allows induction.

The idea here is that you don't need to find a space. It's part of the data of the category. All you have to do is show that it is highly connected. □

Conjecture. *If $(C, \oplus, 0)$ with 0 initial satisfies (H1)-(H2), then (H3) is equivalent to stability.*

Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA 02138
E-mail address: `eriehl@math.harvard.edu`