OPERADS IN ALGEBRAIC TOPOLOGY II

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Contents

Yesterday I gave you a very elementary introduction to operads. The examples I gave you weren't very topological. Today we're going to get into some topology.

The little n-disks operad

Goal. We'll consider how to interpolate "up to homotopy" between As and Com. There is an operad morphism As $\to$ Com.

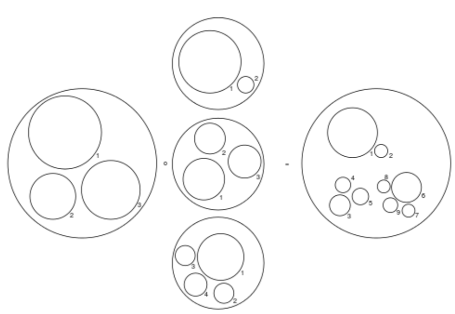
Definition. We'll write $\mathscr{D}_n$ for the *little n-disks operad* (where $(\mathcal{M}, \otimes, I) = (\mathbf{Top}^{\text{nice}}, \times, \{*\})$). Define

$$\mathscr{D}_n(r) = \mathrm{sEmb}(\coprod_r D^n, D^n), \quad D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}.$$

Here sEmb is the space of standard embeddings (a subspace of the mapping space with compact-open topology). This means that on each summand we have a map $D^n \to D^n$ of the form $x \mapsto \lambda x + c$, $0 \leq \lambda \leq 1$. We'll only consider embeddings like this. No rotations are allowed.

An element of $D_2(4)$ looks like a big 2-disk with 4 (disjoint) little 2-disks inside it. Note that Σ_r acts on $\mathscr{D}_n(r)$ by permuting the little disks.

The operad multiplication $\mathscr{D}_n \circ \mathscr{D}_n \to \mathscr{D}_n$ is defined by iterated embeddings of little disks. For example, the following composite is an element of of $\mathscr{D}_2(9)$:



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Remark. There exists an injective operad morphism $\iota_n \colon \mathscr{D}_n \hookrightarrow \mathscr{D}_{n+1}$ given by "equatorial embedding." For $n = 1$, the closed unit interval gets embedded as a diameter (the horizontal, say) of the closed 2-disk. Then any embedded little intervals are embedded as diameters of little 2-disks centered about the equator. So there exists a functor $\iota_n^* \colon \mathbf{Alg}_{\mathscr{D}_{n+1}} \to \mathbf{Alg}_{\mathscr{D}_n}$.

Let $\mathscr{D}_\infty = \mathrm{colim}_n\, \mathscr{D}_n$.

Proposition.

(i) $\mathscr{D}_1 \xrightarrow{\simeq} \mathrm{As}$ *and* $\pi_0\mathscr{D}_1(r) = \Sigma_r$.

(ii) $\mathscr{D}_\infty \xrightarrow{\simeq} \mathrm{Com}$.

Q. What about $\mathscr{D}_n$ for $1 < n < \infty$?

Theorem (May, Boardman-Vogt)**.**

(i) Every n-fold based loop space is a $\mathscr{D}_n$*-algebra.*

(ii) If Y is a $\mathscr{D}_n$*-algebra and if* $\pi_0 Y$ *is a group (and not just a monoid with respect to its induced binary multiplication), then there exists a space X so that* $Y \simeq \Omega^n X$.

Indication of the proof of (i). Observe that $\Omega^n X \cong \mathrm{Map}((D^n, \partial D^n), (X, x_0))$. To show that $\Omega^n X$ is a $\mathscr{D}_n$-algebra, we want

$$\mathscr{D}_n(r) \times \Omega^n X \times \cdots \times \Omega^n X \xrightarrow{\phi_r^\flat} \Omega^n X.$$

Given $d \in \mathscr{D}_n(r)$ and $(\alpha_1, \ldots, \alpha_r)$ a list of r-elements of $\mathrm{Map}((D^n, \partial D^n), (X, x_0))$ the composite $d \circ (\alpha_1, \ldots, \alpha_r)$ is the map of pairs $(D^n, \partial D^n) \to (X, x_0)$ that acts via α_i on disk i and is constant at the basepoint x_0 outside the embedded disks. These actions assemble into a continuous map because the boundary of each embedded disk is also mapped to x_0 via α_i. □

May and Boardman-Vogt have different proofs for (ii). The key observation (in one of the proofs) is that the free $\mathscr{D}_n$-algebra on a space X is $\Omega^n \Sigma^n X$.

Remark. If $n > 1$, $r \geq 1$, then $\mathscr{D}_n(r)$ is path connected. In particular, $\mathscr{D}_n(2)$ is path connected for all $n > 1$. This is the space of binary operations on a $\mathscr{D}_n$-algebra. And thus, there exists a path from any multiplication $m \in \mathscr{D}_n(2)$ to any $m\tau \in \mathscr{D}_n(2)$ (where $\tau \in \Sigma_2$ is the non-identity permutation). This implies that all of the binary mutliplications in a $\mathscr{D}_n$-algebra are homotopy commutative.

Topology of little n-disks

Q. What do the spaces $\mathscr{D}_n(r)$ look like?

Proposition. $\mathscr{D}_n(r) \xrightarrow{\simeq} \mathrm{Conf}(r; \mathring{D}^n)$*, the space of labelled configurations of r points in the open n-disk* $\mathring{D}^n$*. The map takes the embedded disks to their centers.*

In particular, $\mathscr{D}_n(2) \simeq S^{n-1}$. This says that we have a whole sphere's worth of binary multiplications (that are all homotopic).

Remark. If $\mathscr{P}$ is a topological operad, then $H_*\mathscr{P}$ (where $(H_*\mathscr{P})(r) = H_*(\mathscr{P}(r))$) is an operad in graded abelian groups.

The trick is that there is a natural transformation $H_*\mathscr{P}(r) \otimes H_*(\mathscr{P}(t)) \to H_*(\mathscr{P}(r) \times \mathscr{P}(t))$ for all r and t using the fact that homology is a monoidal functor.

Theorem (F. Cohen)**.** *If* $\Bbbk$ *is a field*

(i) $H_*(\mathscr{D}_1; \Bbbk) = \mathrm{As}$
(ii) $H_*(\mathscr{D}_\infty; \Bbbk) = \mathrm{Com}$
(iii) $H_*(\mathscr{D}_n; \Bbbk) = \mathscr{G}_n$*, the n-Gerstenhaber operad.*

A $\mathscr{G}_n$-algebra is an object A with a commutative and associative multiplication $m\colon A \otimes A \to A$ with a bracket $\langle -, - \rangle\colon A \otimes A \to A$ of degree $1 - n$ that satisfies a graded Jacobi identity and anticommutativity. Also, for all $c \in A$, $\langle -, c \rangle\colon A \to A$ is a derivation with respect to $\langle ab, c \rangle = \pm a\langle b, c \rangle \pm \langle a, c \rangle b$.

More generally:

Definition. A *topological operad* $\mathscr{E}$ is an $\mathscr{E}_n$-operad ($1 \leq n \leq \infty$) if it is weakly equivalent (as an operad) to $\mathscr{D}_n$:

$$\mathscr{D}_n \xleftarrow{\simeq} \bullet \xrightarrow{\simeq} \mathscr{E}.$$

Example. The *Barratt-Eccles operad* $\mathscr{E}$ is an $\mathscr{E}_\infty$-operad. Here $\mathscr{E}(r) = E\Sigma_r$ (the total space of the classifying space of the group).

The Deligne Conjecture

$\mathscr{D}_2$ plays a central role in the Deligne conjecture.

Definition. Let A be an associative $\Bbbk$-algebra, for a commutative ring $\Bbbk$. The *Hochschild cochain complex* of A is

$$\mathrm{Hom}(\Bbbk, A) \xrightarrow{d^0} \mathrm{Hom}(A, A) \xrightarrow{d^1} \mathrm{Hom}(A^{\otimes 2}, A) \xrightarrow{d^2} \cdots$$

where $d^n\colon \mathrm{Hom}(A^{\otimes n}, A) \to \mathrm{Hom}(A^{\otimes n+1}, A)$ is given by

$$(d^n f)(a_0 \otimes \cdots \otimes a_n) = a_0 f(a_1 \otimes \cdots \otimes a_n) + \sum_i \pm f(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \pm f(a_0 \otimes \cdots \otimes a_{n-1}) a_n.$$

The notation for this is $C^*(A, A)$. The *Hochschild cohomology of A* is $H^*(A, A) = H^*(C^*(A, A))$. This classifies infinitesimal deformations of the multiplication on A.

Theorem (Gerstenhaber)**.** $H^*(A, A)$ *is a* $\mathscr{G}_2$*-algebra.*

Recall that $\mathscr{G}_2 = H_*\mathscr{D}_2$. Let $\mathscr{S} = C_*\mathscr{D}_2$ (applying chains, not homology), an operad in chain complexes.

Conjecture (Deligne, 1993)**.** *The* $\mathscr{G}_2$*-algebra structure on* $H^*(A, A)$ *lifts (up to homotopy) to an* $\mathscr{S}'$*-algebra structure, where* $\mathscr{S}' \simeq \mathscr{S}$.

There are many partial results and (later) complete proofs. For example, Gerstenhaber-Voronov (1994) and Voronov (1999) build

$$\mathscr{S} \xleftarrow{\simeq} \tilde{\mathscr{S}} \xrightarrow{\simeq} \mathscr{H} \to \mathrm{End}(C^*(A, A))$$

where $\mathscr{H}$ is a homotopy Gerstenhaber operad and $\tilde{\mathscr{S}}$ is a geometric resolution of $\mathscr{S}$.

Also McClure-Smith (1999) show that there is some $\tilde{\mathscr{D}} \simeq \mathscr{D}_2$ so that

$$\mathscr{D} \xleftarrow{\simeq} C_*\tilde{\mathscr{D}} \xrightarrow{\simeq} \mathscr{H} \to \mathrm{End}(C^*(A, A)).$$

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