HOMOTOPY THEORY AND ARITHMETIC GEOMETRY II: GROTHENDIECK'S ANABELIAN CONJECTURES AND $\mathbb{A}^1$-ALGEBRAIC TOPOLOGY

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Introduction

Yesterday we showed that for X a scheme we can introduce $\acute{\mathrm{E}}\mathrm{t}(X)$, a prosimplicial set (an object in $\mathbf{pro}\mathcal{S}$). We say X is over $\Bbbk$ (write $X/\Bbbk$) if the coefficients of the associated polynomials are in $\Bbbk$ and we are only considering solutions $X(R)$ where $\Bbbk \to R$, or more precisely $X \to \mathrm{spec}\Bbbk$.

We have the following properties:

- $X/\mathbb{C}$ then $\acute{\mathrm{E}}\mathrm{t}(X) \simeq X(\mathbb{C})^\wedge$
- $\Bbbk$ a field, the $\acute{\mathrm{E}}\mathrm{t}(\mathrm{spec}\Bbbk) \simeq BG_\Bbbk$ where $G_\Bbbk = \mathrm{Gal}(\Bbbk^s/\Bbbk)$, the Galois group of the separable closure. Then
$$\acute{\mathrm{E}}\mathrm{t}(\mathrm{spec}\Bbbk), \acute{\mathrm{E}}\mathrm{t}(\mathrm{spec}L)] \xrightarrow{=} \mathrm{Hom}^{\mathrm{out,cts}}(G_\Bbbk, G_L).$$
- For $X/\Bbbk$, then $X_{\overline{\Bbbk}} = \mathrm{spec}\overline{\Bbbk} \times_{\mathrm{spec}\Bbbk} X$ and there is a fiber sequence
$$\acute{\mathrm{E}}\mathrm{t}(X_{\overline{\Bbbk}}) \to \acute{\mathrm{E}}\mathrm{t}(X) \to \acute{\mathrm{E}}\mathrm{t}(\mathrm{spec}\Bbbk).$$
- If X is normal, then $\pi_1 \acute{\mathrm{E}}\mathrm{t}(X) = \pi_1^{SGA}(X) = \pi_1 X$.

Grothendieck's anabelian conjectures

Q. Choose $a, b \in \mathbb{Q}^*$. When is $G_{\mathbb{Q}[\sqrt{a}]} \cong G_{\mathbb{Q}[\sqrt{b}]}$?

The answer is lovely: it's if and only if $\mathbb{Q}[\sqrt{a}] \cong \mathbb{Q}[\sqrt{b}]$, which is true if and only if $a = b \in \mathbb{Q}^*/(\mathbb{Q}^*)^2$.

So we conclude that
$$\mathrm{Iso}_{\mathrm{sch}}(\mathrm{spec}\mathbb{Q}[\sqrt{a}], \mathrm{spec}\mathbb{Q}[\sqrt{b}]) \xrightarrow{=} \mathrm{Iso}_{\mathrm{ho(pro}\mathcal{S})}(\acute{\mathrm{E}}\mathrm{t}(\mathrm{spec}\mathbb{Q}[\sqrt{a}]), \acute{\mathrm{E}}\mathrm{t}(\mathrm{spec}\mathbb{Q}[\sqrt{b}])).$$

Theorem (Neukirch-Uchida)**.** *For K, L finite extensions of $\mathbb{Q}$,*
$$\mathrm{Iso}(K, L) \xrightarrow{=} \mathrm{Iso}^{\mathrm{cts,out}}(G_K, G_L).$$

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In Grothendieck's set-up, this property is viewed as a property of the fundamental groups. Recall that here the absolute Galois groups are the fundamental groups. Grothendieck's anabelian conjectures predict a large family of schemes that are controlled by their fundamental groups.

Finite extensions of $\mathbb{Q}$ are called *number fields*.

Conjecture (Grothendieck)**.** *For K a number field, there is a full subcategory $\mathbf{An}_K$ of schemes over K. $\mathbf{An}_K$ includes smooth curves with Euler characteristic $\chi < 0$,* spec$\mathbb{k}$*, and total spaces of fibrations where the bases and fibers are in $\mathbf{An}_K$. For all $X_1, X_2 \in \mathbf{An}_K$, then*

$$\mathrm{Mor}^{\mathrm{dense}}_{\mathrm{sch}}(X_1, X_2) \xrightarrow{=} \mathrm{Mor}^{\mathrm{cts,open}}_{G_K}(\pi_1 X_1, \pi_1 X_2). \tag{1}$$

On the left-hand side we have maps with dense image, and on the right-hand side we have maps with open image.

Theorem (Mochizuki)**.** *If K is a sub p-adic field, X_1 is smooth, and X_2 is smooth with negative Euler characteristic. Then* (1) *holds.*

This provides some evidence for the anabelian conjecture.

The section conjecture

The section conjecture tells us what the solutions to the polynomials are expected to be.

Conjecture (Grothendieck)**.** *Let X be a smooth curve of genus ≥ 2 and compact over K.*[1] *Then the natural map*

$$X(K) \xrightarrow{=} [\acute{E}t(\mathrm{spec} K), \acute{E}t(X)]$$

is a bijection.

The section conjecture is open. There exist curves X, as in the statement of the section conjecture, so that $X(K) = \emptyset$, $[\text{Ét}(\mathrm{spec}K), \text{Ét}(X)] = \emptyset$. It seems strange that the set of homotopy classes of maps could be empty — why not just send everything to a point? The point is that this notation is for maps over Ét(spec$K$). Given

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ & \searrow \quad \swarrow & \\ & \mathrm{spec}K & \end{array}$$

apply Ét to get

$$\begin{array}{ccc} \text{Ét}(X_1) & \longrightarrow & \text{Ét}(X_2) \\ & \searrow \quad \swarrow & \\ & \text{Ét}(\mathrm{spec}K) & \end{array}$$

So it is possible that there are no "sections" of the given map $\text{Ét}(X_2) \to \text{Ét}(\mathrm{spec}K)$. So the section conjecture is true for such X. (Explicitly constructed example due to Stix, Haran-Szamuely, Wittenberg.)

The section conjecture behaves well for finite étale covering spaces. If the section conjecture holds for X and if $Y \to X$ is a finite covering space, then the section conjecture holds for Y.

[1] Compactness is not necessary but without this hypothesis the statement is more complicated.

Model theory. A really lovely piece of work on the section conjecture is actually in model theory, as in the kind of model theory that is going on in the sister program. One can replace $\pi_1 X$ with $G_{K(X)}$ (the Galois group on the rational functions) and state the "birational section conjecture." This is like saying the section conjecture holds as you pass to smaller and smaller classes of Zariski opens.

Koenigsmann showed the birational section conjecture holds for finite extensions of $\mathbb{Q}_p$ using model theory [S].

Relationship to Sullivan's conjecture. Let X be a smooth curve with negative Euler characteristic. We have $\acute{\mathrm{E}}\mathrm{t}(X) = EG_K \times_{G_K} \acute{\mathrm{E}}\mathrm{t}(X_{\overline{K}}) \to \acute{\mathrm{E}}\mathrm{t}(\mathrm{spec} K) = BG_K$. If we had a section ϕ, we could pull it back along $EG_K \to BG_K$ to get a section ϕ'

$$\begin{array}{ccc} \acute{\mathrm{E}}\mathrm{t}(X) = EG_K \times_{G_K} \acute{\mathrm{E}}\mathrm{t}(X_{\overline{K}}) & \longleftarrow & EG_K \times \acute{\mathrm{E}}\mathrm{t}(X_{\overline{K}}) \\ \phi \uparrow \downarrow & & \downarrow \uparrow \phi' \\ \acute{\mathrm{E}}\mathrm{t}(\mathrm{spec} K) = BG_K & \longleftarrow & EG_K \end{array}$$

This gives $\phi' \in \mathrm{Map}(EG_K, \acute{\mathrm{E}}\mathrm{t}(X_{\overline{K}}))^{G_K} = \acute{\mathrm{E}}\mathrm{t}(X_{\overline{K}})^{hG_K}$. This says that

$$[\acute{\mathrm{E}}\mathrm{t}(\mathrm{spec} K), \acute{\mathrm{E}}\mathrm{t}(X)] \xrightarrow{\equiv} \pi_0 \acute{\mathrm{E}}\mathrm{t}(X_{\overline{K}})^{hG_K}.$$

Thus, the section conjecture says that $X(K) \xrightarrow{\equiv} \pi_0 \acute{\mathrm{E}}\mathrm{t}(X_{\overline{K}})^{hG_K}$.

Sullivan's conjecture is a theorem proved by Miller, Carlsson, Lannes. It says that for G a finite p-group, X a finite G-CW complex, the map

$$(X^G)^\wedge_p \to (X^\wedge_p)^{hG}$$

is a weak equivalence.

This leads to a version of the section conjecture over $\mathbb{R}$ which is true and has several different proofs.

Example. Consider $y^2 = f(x) = (x-1)(x-2)(x-3)(x-4)$. There is a map from the scheme $X \to \mathbb{A}^1$ to the affine line given by $(x, y) \mapsto x$. This defines a 2-sheeted cover. Again, we cut slits between 1 and 2 and between 3 and 4 in both the base and in both sheets of the cover. We flip one over before gluing them together along cylinders. This gives a torus minus the two points at infinity. (Note before we needed genus greater than 2, but with Sullivan's conjecture it doesn't matter. The point is solutions are discrete so that fixed points equal homotopy fixed points.)

The goal of this example is see that the map $\pi_0 X(\mathbb{R}) \to \pi_0 X(\mathbb{C})^{hG_{\mathbb{R}}}$ is a bijection. When x is greater than 4, less than 0, or lies between 2 and 3, y^2 is positive. On the torus, these points define two non-intersecting (but homotopic) circles on the torus that enclose the hole formed between the two attached cylinders, one circle mapping down to $(2, 3)$ and the other mapping down to $x \notin [0, 4]$. When we choose a basepoint $b \in X(\mathbb{R})$, we can identify $\pi_0 X(\mathbb{C})^{hG_{\mathbb{R}}} = H^1(G_{\mathbb{R}}, \pi_1 X(\mathbb{C}))$. Given $x \in X(\mathbb{R})$, we can choose path γ from x to the basepoint b. Then define $X(\mathbb{R}) \to H^1(G_{\mathbb{R}}, \pi_1 X(\mathbb{C}))$ by $x \mapsto [g \mapsto \gamma^{-1} g \gamma]$. We see that $\pi_1 X(\mathbb{C}) = \mathbb{Z} \oplus \mathbb{Z}(1)$ and $H^1(G_{\mathbb{R}}, \pi_1 X(\mathbb{C})) = \mathbb{Z}/2$. So the map

$$X(\mathbb{R}) \to H^1(G_{\mathbb{R}}, \mathbb{Z} \oplus \mathbb{Z}(1)) = \mathbb{Z}/2$$

sends $b \mapsto 0$ and everything not in the component of the basepoint to 1.

You'll have to trust me that the example can be worked out. It can even be worked out for arbitrary fields and for elliptic curves. For any field $\Bbbk$, the map $X(\Bbbk) \to [\acute{\mathrm{E}}\mathrm{t}(\mathrm{spec} \Bbbk), \acute{\mathrm{E}}\mathrm{t}(X)]$

can be identified with the Kummer map of the elliptic curve. This example is an elliptic curve whenever you have a $\mathbb{k}$-point. We just worked out the Kummer map over $\mathbb{R}$.

There is a way to rephrase the section conjecture in terms of fixed points and homotopy fixed points [Q].

References

[Q] G. Quick. Existence of rational points as a homotopy limit problem. arXiv:1309.0463 [math.AG].

[S] J. Stix. Rational Points and Arithmetic of Fundamental Groups. Evidence for the section conjecture.

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