HOMOLOGICAL STABILITY FOR FAMILIES OF GROUPS II

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Last time: we were considering a monoidal category $(C, \oplus, 0)$ with 0 initial. We had axioms:

(H1) $\mathrm{Hom}(A, B)$ is acted on transitively by $\mathrm{Aut}_C(B)$.
(H2) $\mathrm{Aut}(A)$ injects into $\mathrm{Aut}(A \oplus B)$ with image $\mathrm{Fix}(B)$.
(H3) For fixed $A, X \in C$, $\tilde{H}_i(S_n(A, X)) = 0$ for some range $i \ll n$.

Then we saw that $H_i(G_n) \xrightarrow{\cong} H_i(G_{n+1})$ for $i \ll n$ for $G_n = \mathrm{Aut}(A \oplus X^{\oplus n})$.

Example. $C = \mathbf{FI}$ the category of finite sets and injections. Then we recover homological stability for $G_n = \Sigma_n$.

Today I want to discuss more examples and see what does and doesn't work.

General linear groups

I'm just going to do $GL_n(\mathbb{R})$ because real vector spaces are easier to think about. The naïve guess is to take $C = \mathbf{fdVec}_{\mathbb{R}}$, finite dimensional vector spaces over $\mathbb{R}$ and injective linear maps. Then $\oplus$ is direct sum and 0 is the zero-dimensional real vector space, which is initial.

For (H1), we want to consider the action of $GL_n(\mathbb{R}) = \mathrm{Aut}(\mathbb{R}^n)$ on $\mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^n)$. This action is transitive because the maps are injective (defining a partial basis).

For (H2), consider $\mathrm{Aut}(\mathbb{R}^k) = GL_k(\mathbb{R}) \hookrightarrow \mathrm{Aut}(\mathbb{R}^k \oplus \mathbb{R}^n) = GL_{k+n}(\mathbb{R})$ by

$$(A) \mapsto \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}.$$

This map is of course injective. But $\mathrm{Fix}(0 \oplus \mathbb{R}^n)$ is the matrices in $GL_{n+k}(\mathbb{R})$ that fix the last n basis vectors. These look like

$$\begin{pmatrix} A & 0 \\ B & I \end{pmatrix}$$

which is bigger than just $GL_n(\mathbb{R})$.

A better category takes C to be finite dimensional vector spaces over $\mathbb{R}$ but maps are *splitted injective linear maps*. Now

$$\mathrm{Hom}_C(\mathbb{R}^k, \mathbb{R}^n) = \{(f, H) \mid f\colon \mathbb{R}^k \hookrightarrow \mathbb{R}^n, H \subset \mathbb{R}^n, \dim H = n - k, \mathbb{R}^n = f(\mathbb{R}^k) \oplus H\}.$$

We still have $G_n = \mathrm{Aut}_C(\mathbb{R}^n) = GL_n(\mathbb{R})$ because there is only one possible complement to the image of a surjective map. Similarly, 0 is still initial because there is only one possible complement to the image of 0.

For (H1), $\mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^n) = \{(f, H)\}$ and the action of $GL_n(\mathbb{R})$ is still transitive.

For (H2), we have an injective map $0 \oplus \mathbb{R}^n \to \mathbb{R}^k \oplus \mathbb{R}^n$ which is the same inclusion but choosing the obvious $\mathbb{R}^k$ as the complement. We're interested in maps that make the

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diagram

$$\begin{tikzcd} 0 \oplus \mathbb{R}^n \arrow[r] \arrow[dr] & \mathbb{R}^k \oplus \mathbb{R}^n \arrow[d] \\ & \mathbb{R}^k \oplus \mathbb{R}^n \end{tikzcd}$$

commute. Maps in the new category C must preserve the complementary subspaces (not fix them, but as a map of pairs), so these have the form

$$\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$$

which is exactly the image of $GL_k(\mathbb{R})$.

For (H3), take $A = 0$ and $X = \mathbb{R}$. Then $S_n(0, \mathbb{R})$ has vertices $\mathrm{Hom}_C(\mathbb{R}, \mathbb{R}^n)$. These are pairs (f, H) where $f\colon \mathbb{R} \hookrightarrow \mathbb{R}^n$ where $f \oplus H = \mathbb{R}^n$. So a vertex is a line with a basis element and also an orthogonal complement.

A p-simplex is $\{(f_0, H_0), \ldots, (f_p, H_p)\}$ so that there exists a "lift" $\mathbb{R}^{p+1} \xrightarrow{(f,H)} \mathbb{R}^n$. The condition is equivalent to saying that $f = f_0 \oplus \cdots \oplus f_p\colon \mathbb{R}^{p+1} \to \mathbb{R}^n$ is injective, which is a linear independence condition. The second part requires that $H = \cap_i H_i$ and $H_i = H \oplus \oplus_{j\neq i} f_j(\mathbb{R})$. This is a very restrictive condition; most things won't be simplices.

The simplicial complex S_n is a split building studied by Charney, who has proved that it is $\frac{n-3}{2}$ connected.

Variants. There are variants: you can replace $\mathbb{R}$ by most rings and finite dimensional vector spaces by finitely generated modules. You might have some different connectivity, but the situation is similar. (Note the category C doesn't use the topology on $\mathbb{R}$. Actually, considering the topology makes things less interesting. These non-compact groups have a compact (orthogonal) subgroup and you can show stability by just using the sphere.)

In particular, for the ring $\mathbb{Z}$, C is the category of finitely generated free abelian groups and splitted injective homomorphisms.

Q. What about taking C to be all finitely generated groups with splitted injective homomorphisms, $\oplus$ the direct product, and 0 the trivial group?

This gives problems already with (H1). We have

$$\mathrm{Hom}_C(A, B) = \{(f, H) \mid f\colon A \hookrightarrow B \supset H, B = f(A) \times H\}.$$

We consider the $\mathrm{Aut}(B)$ action. But the problem is that H and H' each being complements of a monomorphism $A \to B$ does not necessarily imply that H and H' are isomorphic. (H1) is some sort of cancelation property that doesn't hold for maps between finitely generated abelian groups.

But we can restrict to some subcategory:

- right angle Artin groups (RAAG, work in progress with Gandinni)
- finite groups
- recent work of Szymik on automorphisms of nilpotent groups

or some subcategory of groups (graph groups?) where the cancelation property holds.

Q. Does there exist $(C, \oplus, 0)$ satisfying (H1)-(H2) but not (H3)?

Maybe you can construct a badly behaved example so that these simplicial complexes are disconnected or something, but I don't know of such an example.

For $\mathrm{Aut}(F_n)$, the free group on n generators, the corresponding C is the category of finitely generated free groups with splitted injective homomorphisms. Here $\oplus = *$, the free product.

Mapping class groups

I wanted to end with a geometric example that is a bit different. Here I want to look at a category C whose objects are pairs $(M, *)$, where M is a manifold with boundary and $* \in \partial M$ is a basepoint. I like dimensions 2 and 3. Maps are $\pi_0 \mathrm{Emd}_*^{\mathrm{proper}}((M, *), (N, *))$. Here $\oplus$ is connected sum # but respecting the basepoints (gluing them together). The initial unit object is $0 = (D^d, *)$.

For $d = 2$, I'm going to claim that C satisfies (H1) and (H2), but these are not so obvious and I'm going to guess that they do not hold in higher dimensions (greater than three). For (H3), we have to pick objects say $A = (D^2, *)$ and $X = (D^2 \backslash p, *)$ a punctured disk. The complex S_n has vertices $\mathrm{Hom}(X, X^{\oplus} n)$ homotopy classes of proper based embeddings of the disk with one puncture into the disk with n punctures. A complicated one can be defined by squashing the disk with one puncture into a thin ribbon and then snaking it around several of the n punctures before mapping it onto the chosen puncture. So really we see that this is the space of arcs in the based space of a disk with n punctures from the basepoint to a puncture.

The vertices $(f_0, \ldots, f_p)$ form a p-simplex if and only if the corresponding arcs can be made disjoint. Then these assemble (up to homotopy) to a map f from the disk with $p + 1$ punctures to the disk with n punctures such that when I restrict to the arcs I get the original embeddings of the disk with one puncture.

I can't draw what this is as a space, though it's maybe a little less bad than for the vector spaces. This is something I've done in a paper with Allen Hatcher. The point is these spaces are some function of n connected.

You can do the same type of analysis replacing X by some surface with genus.

For dimension 3 manifolds, this is the complex we use to prove homological stability for mapping class groups of 3-manifolds.

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