# Configuration Spaces

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Connections for Women: Alg Top, MSRI

January 24, 2014

## Definition

If $M$ is a smooth, connected, oriented manifold,

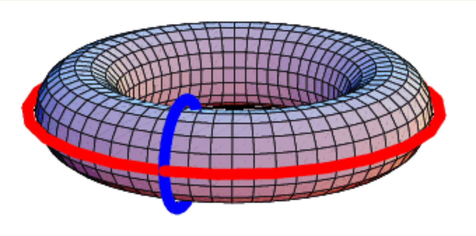
$$\mathrm{Conf}_n(M) = \{(x_1, \ldots, x_n) \in M^n | x_i \neq x_j \text{ if } i \neq j\}$$

$$\overline{\mathrm{Conf}}_n(M) = \mathrm{Conf}_n(M)/\Sigma_n$$

$\overline{\mathrm{Conf}}_n(\mathbb{R}^2) = K(\pi, 1)$ where $\pi = B_n$

Generators of $H_*(M)$ suggest hom classes of $\mathrm{Conf}_n(M), \overline{\mathrm{Conf}}_n(M)$.

## Example



Credit: *http://mathworld.wolfram.com/HomologyIntersection.html*

$\alpha \in H_1(T)$ suggests $[N] \in H_1(\mathrm{Conf}_n(T))$ where $N$=first point travels around $\alpha$, the other points are fixed.

$$H_i(\mathrm{Conf}_2(T)) \cong H^i(\mathrm{Conf}_2(T)) \cong \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}^4 & i = 1 \\ \mathbb{Z}^5 & i = 2 \\ \mathbb{Z}^2 & i = 3 \end{cases}$$

$$H_i(\overline{\mathrm{Conf}}_2(T)) \cong \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}^2 \oplus \mathbb{Z}_2 & i = 1 \\ \mathbb{Z}^3 & i = 2 \\ \mathbb{Z}^2 & i = 3 \end{cases}$$

$$H^i(\overline{\mathrm{Conf}}_2(T)) \cong \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}^2 & i = 1 \\ \mathbb{Z}^3 \oplus \mathbb{Z}_2 & i = 2 \\ \mathbb{Z}^2 & i = 3 \end{cases}$$

Which classes do we expect to appear in $H_*(\mathrm{Conf}_n(M))$ or $H_*(\overline{\mathrm{Conf}}_n(M))$?

- Inherit submanifolds representing classes via embeddings of $S^1 \hookrightarrow M$, $D^2 \hookrightarrow M$
- For $\overline{\mathrm{Conf}}_n(M)$, two points switching places is a 'mod-2' homology class which we expect to appear universally. (What about the 'mod-2' cohom class?)

Little n-disks operad $\mathcal{D}isks^n$ has $\mathcal{D}isks^n(r)$=$r$ disjoint balls in $D^n$

### Proposition

$\mathcal{D}isks^n(r)$ is homotopy equivalent to $\mathrm{Conf}_r(\mathbb{R}^n)$

### Theorem (Sinha, 2010)

*$H_*(\mathcal{D}isks^n) \cong H_*(Conf_*(\mathbb{R}^n) \cong \mathcal{P}ois^n$ as operads via nice geometric operad structure on $H_*(Conf_r(\mathbb{R}^n))$*

## Definition

$\mathcal{P}ois^n$=forests of trees with numbered leaves and anti-symmetry, jacobi identity

## Claim

Systems of orbits represent generators of $H_i(\mathrm{Conf}_r(\mathbb{R}^n))$ and correspond to $r$-arity part of $\mathcal{P}ois^n$

Thank you!


[F. R. Cohen, T. J. Lada, P. J. May]
The homology of iterated loop spaces
Lecture Notes in Mathematics 533
Springer (1976)

E. R. Fadell, S. Y. Husseini
Geometry and Topology of Configuration Spaces
Springer Monographs in Mathematics (2001)

[D. P. Sinha]
The homology of the little disks operad
http://arxiv.org/abs/math/0610236 (2010)

# CONFIGURATION SPACES

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I'm going to give a quick overview of some interesting things to do with configuration spaces $\mathrm{Conf}_n(M)$, for ordered configurations, and $\overline{\mathrm{Conf}}_n(M)$ for unordered configurations. Note $\overline{\mathrm{Conf}}_n(\mathbb{R}^2)$ is a $K(\pi, 1)$ with $\pi$ the braid group.

Generators of $H_*(M)$ suggest homology classes of $\mathrm{Conf}_n(M)$ and $\overline{\mathrm{Conf}}_n(M)$.

For example, I've computed homology and cohomology for $\mathrm{Conf}_2(T)$ and for $\overline{\mathrm{Conf}}_2(T)$. Note there is some 2-torsion for unordered configurations.

Which classes do we expect to appear in or $H_*\mathrm{Conf}_n(M)$ or $H_*\overline{\mathrm{Conf}}_n(M)$? One pattern arises from submanifolds representing classes via the embeddings $S^1 \hookrightarrow M$ and $D^2 \hookrightarrow M$. For unordered configurations, we expect the mod-2 homology classes that comes from two points switching places to appear universally.

**Proposition.** *The little n-disks operad* $\mathrm{Disks}_n(r)$ *is homotopy equivalent to* $\mathrm{Conf}_r(\mathbb{R}^n)$.

**Theorem** (Sinha). $H_*(\mathrm{Disks}_n) \cong H_*(\mathrm{Conf}_*(\mathbb{R}^n)) \cong \mathrm{Pois}_n$*, the Poisson operad, as operads via nice geometric operad structure on the homology of configurations of r points in* $\mathbb{R}^n$.

Define the Poisson operad $\mathrm{Pois}_n$ to be forests of trees with numbered leaves and anti-symmetry and jacobi identities. We claim that the system of orbits represent generators of $H_i(\mathrm{Conf}_r(\mathbb{R}^n))$, which correspond to the $r$-arity part of $\mathrm{Pois}_n$.

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*Date*: Connections for Women: Algebraic Topology — MSRI — 24 January, 2014.