An equivariant infinite loop space machine

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Classical infinite loop space machines

Infinite loop space machines: turn algebraic/categorical data into spectra.

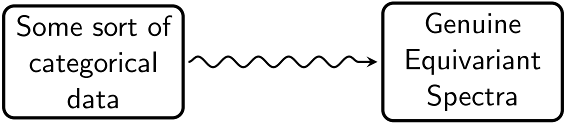
$$\boxed{\begin{array}{c}\text{Permutative}\\ \text{Categories}\end{array}} \rightsquigarrow \boxed{\begin{array}{c}\text{Infinite}\\ \text{Loop Spaces}\end{array}} \rightsquigarrow \boxed{\begin{array}{c}\text{Connective}\\ \text{Spectra}\end{array}}$$

Give Eilenberg–Mac Lane spectra, K-theory, Thom spectra

Many varieties: E_∞ operads (May), Γ-spaces (Segal)...

Equivariant infinite loop space machines

Goal: Build a machine that produces *genuine equivariant* spectra from appropriate categorical/algebraic data.



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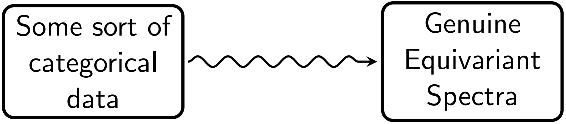


Definition

A *genuine G-equivariant* spectrum has deloopings for all finite dimensional real G representations.

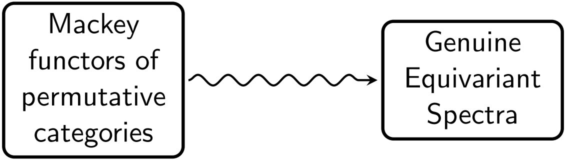
Lewis–May–Steinberger; Costenoble–Waner; Shimakawa; work in progress by May–Merling–Osorno, Guillou–May

Main Theorem

Let G be a finite group.

Main Theorem (Bohmann–Osorno)

There is an equivariant infinite loop space machine that produces genuine G-equivariant spectra from a "Mackey functor of permutative categories".



Mackey functors

Definition

The *Burnside category* of G is a category with

- objects: finite G-sets
- morphisms: (group completion of isomorphism classes of) spans of finite G-sets

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

Definition

A *Mackey functor* is an additive functor from the Burnside category to abelian groups.

Homotopy Mackey functors

Homotopy groups of genuine G-spectra are given by *Mackey functors*.

For G-spaces:

$$G/H \longmapsto \pi_n(X^H)$$

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For G-spectra:

- Get additional map $\pi_n(X^H) \to \pi_n(X^K)$
- Encoded as a Mackey functor $\underline{\pi}_n(X)(-)$

Mackey structure for genuine G-spectra

"Mackey functor" shape captures the essence of genuine G-spectra.

Theorem [Guillou–May]

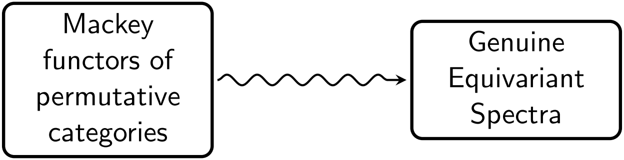
For finite G, the category of genuine G-spectra is Quillen equivalent to the category of "spectrally-enriched Mackey functors of spectra".

"Spectrally-enriched Mackey functors of spectra:"

1. Build combinatorial Burnside 2-category $\mathcal{E}G$
2. Spectrally enrich via K-theory to get $\mathbb{K}\mathcal{E}G$
3. Take functors $\mathbb{K}\mathcal{E}G \to \mathrm{Spec}$ into (nonequivariant) spectra

Components of the equivariant machine

Our machine:



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Components of the equivariant machine

Our machine:

$$\boxed{\text{Mackey functors of permutative categories}} \rightsquigarrow \boxed{\text{Spectral functors } \mathbb{K}\mathcal{E}G \to \text{Spec}}$$

Definition

A Mackey functor of permutative categories is a “permutative category enriched” functor from the combinatorial Burnside 2-category to a 2-category of permutative categories.

PC-category

"Definition"

A **PC**-*category* is a "category enriched in permutative categories."

$\mathcal{C}$ consists of

- collection of objects
- $\mathcal{C}(X, Y)$ permutative category
- composition $\mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$ is a bilinear functor

Applying K-theory to morphisms gives a spectrally enriched category $\mathbb{K}\mathcal{C}$.

Combinatorial Burnside 2-category

Definition

The combinatorial Burnside 2-category $\mathcal{E}G$ is a **PC**-category with

- objects: finite G-sets X, Y
- morphisms $\mathcal{E}G(X, Y)$: permutative category of spans of finite G-sets

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

  under disjoint union

Usual Burnside category is a quotient of $\mathcal{E}G$.

The categorical input

Definition

Let Perm be the **PC**-category with

- objects: permutative categories $\mathcal{A}, \mathcal{B}$
- morphisms $\mathrm{Perm}(\mathcal{A}, \mathcal{B})$: category of symmetric monoidal functors $\mathcal{A} \to \mathcal{B}$ and strictly unital natural transformations

Mackey functors of permutative categories are functors $\mathcal{E}G \to \mathrm{Perm}$ preserving the **PC**-category structure.

Key ingredients

- Guillou–May Theorem
- A good K-theory machine: behaves well with respect to bilinear maps
- A lemma:

Lemma

There is a spectrally enriched functor $\Phi\colon \mathbb{K}\mathrm{Perm} \to \mathrm{Spec}$ that takes a permutative category $\mathcal{A}$ to the K-theory spectrum of $\mathcal{A}$.

The machine goes!

How do we turn a functor $\mathcal{E}G \to \text{Perm}$ into a functor $\mathbb{K}\mathcal{E}G \to \text{Spec}$?

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\mathbb{K}\mathcal{E}G \to \mathbb{K}\mathrm{Perm} \xrightarrow{\Phi} \mathrm{Spec}
\end{tikzcd}$$

Examples

- Eilenberg–Mac Lane spectra for Mackey functors

- Connective topological K-theory

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$$\mathcal{B}G \xrightarrow{M} \mathrm{Ab}$$

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- Eilenberg–Mac Lane spectra for Mackey functors

$$\mathcal{E}G \longrightarrow \mathcal{B}G \xrightarrow{\quad M \quad} \mathrm{Ab} \longrightarrow \mathrm{Perm}$$

- Connective topological K-theory

AN EQUIVARIANT INFINITE LOOP SPACE MACHINE

ANNA MARIE BOHMANN AND ANGÉLICA OSORNO

Ordinary infinite loop space machine to equivariant machines

Infinite loop space machines turn algebraic/categorical data into spectra. There are a variety of such. They take permutative categories (strict symmetric monoidal categories) as input and produce Eilenberg-MacLane spectra, K-theory, Thom spectra.

The goal for an *equivariant* infinite loop space machine is to produce a *genuine equivariant* spectrum: this has deloopings for all finite dimensional real G representation spheres (G a finite group). Given a G-representation V with group acting, take the 1-point compactification to get a representation sphere S^V. There is previous work by Lewis-May-Steinberger, Costenoble-Waner, and Shimakawa and work-in-progress to do this by May-Merling-Osorno and Guillou-May. The problem is these machines don't give good control over the fixed points. Our machine very directly gives this fixed point information.

Theorem (Bohmann-Osorno)**.** *There is an equivariant infinite loop space machine that produces genuine G-equivariant spectra from a "Mackey functor of permutative categories."*

Mackey functors

Definition. The *Burnside category* of G is a category whose objects are finite G-sets and whose morphism are (group completions of isomorphism classes of) spans of finite G-sets $X \leftarrow Z \rightarrow Y$. A *Mackey functor* is an additive functor from the Burnside category to abelian groups.

Why are Mackey functors important? Homotopy groups of genuine G-spectra are given by Mackey functors. For G-spaces:

$$G/H \mapsto \pi_n(X^H)$$

and

$$\begin{tikzcd}
G/H \arrow[r, mapsto] \arrow[d] \arrow[dr, phantom, "\mapsto"] & \pi_n(X^H) \\
G/K \arrow[r, mapsto] & \pi_n(X^K) \arrow[u]
\end{tikzcd}$$

For G-spectra, you get an additional map $\pi_n(X^H) \rightarrow \pi_n(X^K)$. All this information is encoded as a Mackey functor $\underline{\pi}_n(X)(-)$.

"Mackey functor" shape categories capture the essence of genuine G-spectra.

Theorem (Guillou-May)**.** *For finite G, the category of genuine G-spectra is Quillen equivalent to the category of "spectrally-enriched Mackey functors of spectra."*

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Using this equivalence, our machine really produces spectrally-enriched Mackey functors of spectra. Now I really owe you the definition of a Mackey functor of permutative categories.

Mackey functors of permutative categories

Definition. A *PC-category* is a "category enriched in permutative categories." This is a (strict) 2-category C with extra structure: $C(X,Y)$ is a permutative category and composition $C(Y,Z) \times C(X,Y) \to C(X,Z)$ is a bilinear functor. A *PC-functor* is an enriched functor of permutative categories.

Applying a good K-theory machine (e.g., Elmendorf-Mandell) to the morphism categories gives a spectrally enriched category $\mathbb{K}C$.

Definition. The *combinatorial Burnside 2-category* $\mathcal{E}G$ is a PC-category with objects finite G-sets and morphisms $\mathcal{E}G(X,Y)$ the permutative category of spans of finite G-sets under disjoint union. The usual Burnside category $\mathcal{B}G$ is a quotient of $\mathcal{E}G$.

Definition. Let **Perm** be the PC-category with objects permutative categories and morphisms $\mathbf{Perm}(\mathcal{A},\mathcal{B})$ the category of symmetric monoidal functors and strictly unital natural transformations.

Mackey functors of permutative categories are PC-functors $\mathcal{E}G \to \mathbf{Perm}$.

The machine

The key ingredients are the Guillou-May theorem, a good K-theory machine (behaves well with respect to bilinear maps), and a lemma:

Lemma. *There is a spectrally enriched functor $\phi\colon \mathbb{K}\mathbf{Perm} \to \mathbf{Spec}$ that takes a permutative category $\mathcal{A}$ to the K-theory spectrum of $\mathcal{A}$.*

Start with a PC-functor $\mathcal{E}G \to \mathbf{Perm}$, apply $\mathbb{K}$ to get $\mathbb{K}\mathcal{E}G \to \mathbb{K}\mathbf{Perm}$, then postcompose to get $\mathbb{K}\mathcal{E}G \to \mathbb{K}\mathbf{Perm} \to \mathbf{Spec}$.

Example. Eilenberg-MacLane spectra for Mackey functors. Given a Mackey functor $M\colon \mathcal{B}G \to \mathbf{Ab}$ we can get a PC-functor $\mathcal{E}G \to \mathcal{B}G \xrightarrow{M} \mathbf{Ab} \to \mathbf{Perm}$.

Example. connective topological K-theory

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