Schematic Homotopy Types of Operads

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Background

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- Morphisms in $\mathcal{C}$ are $\mathbf{k}$-morphisms between these objects.

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- Morphisms in $\mathcal{C}$ are $\mathbf{k}$-morphisms between these objects.
- As an example take $\mathcal{C}$ to be the category of (finite type) $\mathbf{k}$ schemes and $\mathbf{k}$ morphisms.
- Let $X \in \mathcal{C}$, then $\pi_1^{geom}(X)$ denotes the "geometric" fundamental group of X, i.e. $\pi_1(X \otimes \bar{\mathbf{k}})$

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- It is well known that $\pi_1^{geom}(X)$ is a finitely generated profinite group.
- Therefore, $\pi_1^{geom}(-)$ is a functor from $\mathcal{C}$ to the category of (topologically) finitely generated profinite groups.
- The (outer) automorphisms of this functor form group denoted by $\mathbf{Out}(\pi_1^{geom}(\mathcal{C}))$.

Background

- If our outer automorphisms satisfy some Galois-style properties then we have a canonical outer action of $\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k})$ on $\pi_1^{geom}(-)$ which is compatible with morphisms.
- This gives a homomorphism

$$\varphi : \mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k}) \longrightarrow \mathbf{Out}(\pi_1^{geom}(\mathcal{C}))$$

which is injective in all the interesting cases.

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- Fresse recently showed that, over $\mathbb{Q}$, a profinite version of $\mathbf{Out}(\pi_1^{geom}(\mathcal{C}))$ is isomorphic to homotopy automorphisms of D_2.
- A key component to Fresse's computation is the development of a **rationalization of an operad**.

Theorem (Fresse)

There exists a rationalization of the little 2-disks operad.

Schematization

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- The sheaf of groups $\pi_1((X \otimes \mathbf{k})^{sch}, x)$ is *represented* by the *proalgebraic completion* of the discrete group $\pi_1(X, x)$.
- There are functorial isomorphisms $H_n((X \otimes \mathbf{k})^{sch}, \mathbb{G}_a) \cong H_n(X, \mathbf{k})$.
- If X is a simply connected finite CW-complex, there exist functorial isomorphisms $\pi_i((X \otimes \mathbf{k})^{sch}, x) \cong \pi_i(X, x) \otimes \mathbb{G}_a$.
- For a connected nilpotent X and $\mathbf{k} = \mathbb{Q}$ (respectively $k = \mathbb{F}_p$) the object $(X \otimes k)^{sch}$ is a model for the rational (respectively p-adic) homotopy type of X.

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- The geometric spectrum of a commutative algebra

$$\mathrm{Spec} : (\mathrm{Alg})^{op} \to \mathrm{Sch}_{\mathbb{C}}$$

  gives us schemes.
- This is an equivalence of categories.
- The left adjoint is denoted by O.

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- To construct $(X \otimes \mathbf{k})^{sch}$ we make use of a "derived" version of the functors Spec and O. We have to replace Alg and $\mathrm{Sch}_{\mathbb{C}}$ with model categories.
- Let Alg^{Δ} be the category of unital commutative cosimplical $\mathbb{C}$-algebras.
- Alg^{Δ} is a simplicial monoidal model category with weak equivalences the quasi-isomorphisms (epimorphisms) of the associated normalized chain complexes.

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- The category of $\mathrm{Sch}_{\mathbb{C}}$ sits inside $\mathrm{sPr}_*(\mathbb{C})$
- $\mathrm{sPr}_*(\mathbb{C})$ has a simplicial, monoidal model category structure.

Schematization

- We define the geometric spectrum of a co-simplicial algebra

$$\mathrm{Spec} : (\mathrm{Alg}_{\mathbb{C}}^{\Delta})^{op} \to \mathrm{sPr}_*(\mathbb{C})$$

  by

- The presheaf of n-simplices of $\mathrm{Spec}(A)$ is given by

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Schematization

Theorem (Toën)

The functors $Spec$ and O form a Quillen adjoint pair, i.e.

$$\mathbb{R}Spec : Ho(Alg^{\Delta}) \rightleftarrows Ho(sPr_*) : \mathbb{L}O$$

is well defined.

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- Let F be a pointed simplicial presheaf.
- The morphism $F \to \mathbb{R}Spec\mathbb{L}O(F)$ is universal among morphisms from F to (equivariant) affine stacks. The stack $\mathbb{R}\mathrm{Spec}\mathbb{L}O(F)$ is then called an affinization (schematization) over $\mathbb{C}$ of the stack F and is denoted by $(F \otimes \mathbb{C})^{uni}$ $((F \otimes \mathbb{C})^{sch})$.

Schematization

Theorem (Toën)

Any pointed, connected simplicial set (X, x) possesses a schematization over $\mathbb{C}$.

Schematization of Operads

- An **operad** is a graded object $P = \{P(n)\}_{n \geq 0}$ together with a (highly) non-commutative multiplication

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- A **co-operad** is a graded object $P = \{P(n)\}_{n \geq 0}$ together with a non-commutative co-multiplication

$$P(n+m-1) \longrightarrow P(n) \circ P(m).$$

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Definition (R)

Given a $P = \{P(n)\}_{n\geq 0} \in Op_{01}(\mathrm{sPr}_*)$, then $\mathbb{R}Spec\mathbb{L}O(P) = \{\mathbb{R}Spec\mathbb{L}O(P(n))\}_{n\geq 0}$ is a schematization of P.

Schematization of Operads-The Main Difficulty

- The functor
$$O : \mathrm{sPr}_*(\mathbb{C}) \to \mathrm{Alg}^{\Delta}$$
is adjoint to Spec
- We have a natural product
$$\mu : O(X) \otimes O(Y) \longrightarrow O(X \times Y)$$
which induces the Künneth morphism at the (reduced, sheaf) cohomology level.
- This map is a weak equivalence if X and Y satisfy some conditions but is **not** an isomorphism in general.

- So given (co)operad P in sPr_* we have a chain of cosimplical algebra morphisms

$$O(P(m+n-1)) \xrightarrow{\circ_k^*} O(P(m) \times P(n)) \leftarrow O(P(m)) \otimes O(P(n))$$

  associated to the cooperad $P(m+n-1) \to P(m) \circ P(n)$.

- This assembles into $O(P) = \{O(P(n))\}$ with a **weak homotopy** comultiplication.

- So we must rigidify to get a cooperad structure on $O(P)$.

Existence of Schematization

Theorem (R)

For the operad of little 2-discs, we get a zig-zag of operad homomorphisms:

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- The existence of the map $\bullet \leftarrow E_2$ is equivalent to the existence of an associator over $\mathbb{C}$.
- Drinfeld shows that $\mathrm{Ass}_{\mathbb{C}} \neq \emptyset$.

SCHEMATIC HOMOTOPY TYPES OF OPERADS

MARCY ROBERTSON

Introduction

The goal of this talk is to define the words in the title. Let C be a category whose objects are (finite type) geometric objects over a field $\Bbbk$. An example is the category of finite type schemes over $\Bbbk$. For $X \in C$ then there is a geometric fundamental group $\pi_1^{\text{geom}}(X)$ which is $\pi_1(X \otimes \Bbbk)$. This π_1^{geom} defines a functor from C to the category of finitely generated profintie groups. The outer automorphisms form a group $\text{Out}(\pi_1^{\text{geom}}(C)$. Under nice properties we have a homomorphism

$$\phi\colon \text{Gal}(\overline{\Bbbk}/\Bbbk) \to \text{Out}(\pi_1^{\text{geom}}(C)$$

which is injective in the interesting cases. This group of outer automorphisms is called the *Grothendieck-Teichmuller group*.

I want to generalize a result of Fresse, which shows that over $\mathbb{Q}$, a profinite version of this $\text{Out}(\pi_1^{\text{geom}}(C)$ is isomorphic to the homotopy automorphisms of the little 2-disks operad D_2. A key component is the development of the *rationalization of an operad*.

Theorem (Fresse)**.** *There exists a rationalization of the little 2-disks operad.*

Schematization

For any pointed connected space X and field $\Bbbk$ the *schematization* of X is $(X \otimes \Bbbk)^{\text{sch}}$ is the "closest affine stack to X." The sheaf of groups $\pi_1((X \otimes \Bbbk)^{\text{sch}})$ is represented by the pro algebraic completion of the discrete group $\pi_1 X$. For X connected nilpotent and $\Bbbk = \mathbb{Q}$ (or $\mathbb{F}_p$), then $(X \otimes \Bbbk)^{\text{sch}}$ is a model for the rational (or p-adic) homotopy type of X.

For the rest of the talk fix $\Bbbk = \mathbb{C}$ and let **Alg** denote commutative, unital $\mathbb{C}$-algebras, and $\mathbf{Sch}_{\mathbb{C}}$ the category of affine schemes.

The geometric spectrum of a commutative algebra is a scheme. There is a functor

$$\text{Spec}\colon \mathbf{Alg}^{\text{op}} \to \mathbf{Sch}_{\mathbb{C}}.$$

There is a left adjoint $\mathcal{O}$ (global sections). I'm going to derive this adjunction using model categories.

We have $\mathbf{Alg}^{\Delta}$, cosimplicial algebras, a simplicial monoidal model category. Let $\mathbf{sPr}_*(\mathbb{C})$ denote the category of pointed simplicial presheaves on $\mathbf{Sch}_{\mathbb{C}}^{ffqc}$. The category $\mathbf{Sch}_{\mathbb{C}}$ sits inside this via the Yoneda embedding. Again $\mathbf{sPr}_*(\mathbb{C})$ is a simplicial monoidal model category.

We define the geometric spectrum of a cosimplicial algebra

$$\text{Spec}\colon (\mathbf{Alg}_{\mathbb{C}}^{\Delta})^{\text{op}} \to \mathbf{sPr}_*(\mathbb{C}).$$

Again this has a left adjoint $\mathcal{O}$.

Theorem (Toën)**.** *The functors* Spec *and* $\mathcal{O}$ *form a Quillen adjoint pair. This gives an adjunction (taking derived functors) between their homotopy categories.*

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An *affine stack* is an object F in $\mathrm{Ho}(\mathbf{sPr}_*(\mathbb{C}))$ in the image of the derived functor of Spec. For any pointed simplicial presheaf F, the counit of the derived adjunction is universal among morphisms of F to (equivariant) affine stacks. The stack $\mathbb{R}\mathrm{Spec}\mathbb{L}\mathcal{O}(F)$ is then called the *affineatization*.

Theorem (Toën)**.** *Any pointed connected simplicial set possesses a schematization over* $\mathbb{C}$.

An *operad* is a graded object $\{P(n)\}_{n\geq 0}$ with a multiplication $P(n)\circ P(m) \to P(n+m-1)$. A *co-operad* is a graded object with co-multiplication $P(n+m-1) \to P(n)\circ P(m)$.

Write $\mathbf{Op}_{01}$ for connected co-operads. We show that $\mathbf{Op}_{01}(\mathbf{sPr}_*)$ and $\mathbf{HopfOp}_{01}^{\Delta}$ admit a good notion of homotopy theory and there is a $\mathcal{O} \dashv$ Spec Quillen adjunction on these categories of comonoids.

Definition (R)**.** Given $P \in \mathbf{Op}_{01}(\mathbf{sPr}_*)$ then $\mathbb{R}\mathrm{Spec}\mathbb{L}\mathcal{O}P$ is a schematization of the operad P.

The problem is that the left adjoint $\mathcal{O}$ is not strong monoidal. The map induced by Künneth is a weak equivalence under some conditions but not an isomorphism. So we have to rigidify to get a co-operad structure

Theorem (R)**.** *For the operad of little 2-disks, we get a zig-zag of operad homomorphisms:*

$$\mathbb{R}\mathrm{Spec}H^*(\mathbb{L}\mathcal{O}(D_2)) \to \bullet \leftarrow \bullet \leftarrow D_2$$

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