Realizability of G-modules: on a dual of a Steenrod Problem

Cristina Costoya
(joint with Antonio Viruel)

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  Is there X such that $H_k(X, \mathbb{Z}) \cong G$?

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Example 2 (Steenrod'60, G-Moore spaces problem)

- G group acting on a finitely generated $\mathbb{Z}$-module M.
- $H_*(-, \mathbb{Z})$ homology concentrated on a degree $k \geq 2$

  Is there a G-space X such that $H_k(X, \mathbb{Z}) \cong M$ as $\mathbb{Z}G$-modules?

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$G \cong \mathcal{E}(X)$ for some X?

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Which finite groups are realizable by simply connected rational spaces?

New perspective

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Idea. Introduce graphs on the picture:

$$\begin{aligned} \text{groups} &\longrightarrow \text{graphs} \\ \text{graphs} &\longrightarrow \text{CDGA's} \\ \text{CDGA's} &\longrightarrow \text{rational homotopy types} \end{aligned}$$

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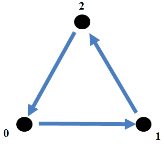
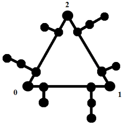
Example ($G = \mathbb{Z}_3$; Cayley graph $\to$ simple graph)





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Our problem (revisited)

Let $\mathcal{G} = (V, E)$ be a finite, simple, connected graph (with more than one vertex). Does there exist a space X such that $\mathrm{Aut}(\mathcal{G}) \cong \mathcal{E}(X)$?

Techniques

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- $\mathcal{G} = (V, E)$, $|V| > 1$
- $f : \mathcal{G}_1 \hookrightarrow \mathcal{G}_2$ such that $[v, w]$ edge of $\mathcal{G}_1$ iff $[f(v), f(w)]$ edge of $\mathcal{G}_2$

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(based on an example of Arkowitz-Lupton)

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- generators in dimensions: $|x_1| = 8, |x_2| = 10, |y_1| = 33, |y_2| = 35,$ $|y_3| = 37, |z| = 119, |x_v| = 40, |z_v| = 119,$

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What happens if G acts on a $\mathbb{Z}$-module M?

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- The field of fractions $\mathbb{Q}(V)$ is a Galois extension of $\mathbb{Q}(V)^G$ with Galois group G.
- $\mathbb{Q}[V]^G$ is finitely generated.

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Corollary (Characterization of finite $G < GL(V)$)

There exists $p_1, \ldots, p_r \in \mathbb{Q}[V]$ such that, for $f \in GL(V)$

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- $q_0 = \sum_1^N \lambda_j v_j^2$, for a good choice of basis of V^* ($N = dim_{\mathbb{Q}} V$, $\lambda_j \neq 0, \forall j$).

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There exist algebraic forms $q_0, q_1, \dots q_r \in \mathbb{Q}[V]^G$ such that:

- $q_0 = \sum_1^N \lambda_j v_j^2$, for a good choice of basis of V^* ($N = dim_{\mathbb{Q}} V$, $\lambda_j \neq 0, \forall j$).
- $deg(q_i) < \deg(q_{i+1})$ for all i.

Realizability of actions: techniques

Corollary (Characterization of finite $G < GL(V)$)

There exists $p_1, \ldots, p_r \in \mathbb{Q}[V]$ such that, for $f \in GL(V)$

$$f \in G \quad \text{if and only if} \quad fp_i = p_i,\ i = 1, \ldots, r.$$

we modify those algebraic forms

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- $deg(q_i) < \deg(q_{i+1})$ for all i.
- For $f \in GL(V)$, $f \in G$ if and only if $fq_i = q_i$, for all i. Therefore

$$G = O(q_0, q_1, \ldots, q_r).$$

Realizability of actions: techniques

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Realizability of actions: techniques

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Realizability of actions: techniques

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□

Fix an integer n such that $\deg(q_r) < 2n + 1$ and define

Realizability of actions: result

$$\mathcal{M}_n = \Big(\Lambda(x_1, x_2, y_1, y_2, y_3, z, v_j \mid j = 1, \ldots, N), d\Big)$$

$$\begin{aligned}
\deg x_1 &= 8, & d(x_1) &= 0\\
\deg x_2 &= 10, & d(x_2) &= 0\\
\deg y_1 &= 33, & d(y_1) &= x_1^3 x_2\\
\deg y_2 &= 35, & d(y_2) &= x_1^2 x_2^2\\
\deg y_3 &= 37, & d(y_3) &= x_1 x_2^3\\
\deg v_j &= 40, & d(v_j) &= 0\\
\deg z &= 80n + 39, & d(z) &= \sum_{i=1}^{r} q_i x_1^{10n+5-5\deg(q_i)} + q_0(x_1^{10n-5} + x_2^{8n-4})\\
& & &\quad + x_1^{10(n-1)}(y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6)\\
& & &\quad + x_1^{10n+5} + x_2^{8n+4}.
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Codifies the G action.

Realizability of actions: result

Theorem (C.-Viruel)

Let G be a finite group, and V a finitely generated faithful $\mathbb{Q}G$-module. Then, there exists infinitely many (non homotopically equivalent) Postnikov pieces X such that, for some $k \geq 2$,

$$(G, V) \cong (\mathcal{E}(X), \pi_k X).$$

Question

Both "Graphs" and "Ring of Invariants" constructions are based on the Arkowitz-Lupton homotopically rigid algebra:

$$\mathcal{M} = \Big(\Lambda(x_1, x_2, y_1, y_2, y_3, z), d\Big)$$

$$\begin{aligned}
\deg x_1 &= 8, & d(x_1) &= 0\\
\deg x_2 &= 10, & d(x_2) &= 0\\
\deg y_1 &= 33, & d(y_1) &= x_1^3 x_2\\
\deg y_2 &= 35, & d(y_2) &= x_1^2 x_2^2\\
\deg y_3 &= 37, & d(y_3) &= x_1 x_2^3\\
\deg z &= 119, & d(z) &= y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6\\
& & &\quad + x_1^{15} + x_2^{12}.
\end{aligned}$$

Are there other possible algebras that can be used?

Answer

Fix an even integer $k > 4$, and define

$$\mathcal{M}_k = \Big(\Lambda(x_1, x_2, y_1, y_2, y_3, z), d\Big)$$

$$\begin{aligned}
\deg x_1 &= 5k - 2, & d(x_1) &= 0 \\
\deg x_2 &= 6k - 2, & d(x_2) &= 0 \\
\deg y_1 &= 21k - 9, & d(y_1) &= x_1^3 x_2 \\
\deg y_2 &= 22k - 9, & d(y_2) &= x_1^2 x_2^2 \\
\deg y_3 &= 23k - 9, & d(y_3) &= x_1 x_2^3 \\
\deg z &= 15k^2 - 11k + 1, & d(z) &= x_1^{3k-12}(x_1^2 y_2 y_3 - x_1 x_2 y_1 y_3 + x_2^2 y_1 y_2) \\
&& &\quad + x_1^{\frac{6k-2}{2}} + x_2^{\frac{5k-2}{2}}.
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& & &\quad + x_1^{\frac{6k-2}{2}} + x_2^{\frac{5k-2}{2}}.
\end{aligned}$$

(C.-Viruel) $[\mathcal{M}_k, \mathcal{M}_k] = \{0, 1\}$.

Yes, there are infinitely many highly connected homotopically rigid algebras.

Thank you!

- C.–Viruel, Every finite group is the group of self-homotopy equivalences of an elliptic space. *To appear in Acta Mathematica. (arXiv:1106.1087)*.
- C.–Viruel, Faithful actions on Differential Graded Algebras and the Group Isomorphism Problem. *To appear in Q. J. Math. (DOI: 10.1093/qmath/hat052)*.
- C.–Viruel, Realizability of G-modules: on a dual of a Steenrod problem. *Preprint*.

REALIZABILITY OF G-MODULES: ON A DUAL OF A STEENROD PROBLEM

CRISTINA COSTOYA

Joint with Antonio Viruel.

Realizability problems: Given an algebra structure A and given a homotopy invariant $I(-)$, find a space X such that $I(X) \cong A$.

Example (Moore spaces)**.** G abstract group, $H_*(-, \mathbb{Z})$ homology concentrationed on a degree $k \geq 2$, is there X such that $H_k(X, \mathbb{Z}) \cong G$?

Example (Steenrod)**.** G group acting on a finitely generated $\mathbb{Z}$-module M. Is there a G-space X such that $H_k(X, \mathbb{Z}) \cong M$?

Our problem: let $\mathcal{E}(X)$ be the group of homotopy classes of self homotopy equivalences of X. For an abstract group G, is there a space X so that $\mathcal{E}(X) \cong G$? There is no known general procedure to solve this problem. Only a few cases: $G = \mathrm{Aut}(\pi)$ for a group π (then $X = K(\pi, n)$).

Q. Which finite groups are realizable by simply connected rational spaces?

Note there is a simply connected space X with $\mathcal{E}(X) = \mathbb{Z}/2$.

New perspective

We'll introduce graphs into the picture and move from groups to graphs, graphs to CDGAs, and CDGAs to rational homotopy types.

Theorem (Frucht '39)**.** *Every finite group G is realizable by a finite, connected, simple graph $\mathcal{G}$ with $G \cong \mathrm{Aut}(\mathcal{G})$.*

Example. For $G = \mathbb{Z}/3$, replace the Cayley graph by a simple (non-directed) graph.

The problem, revisited. Our problem is now for $\mathcal{G}$ a finite, connected, simple graph in place of the group G above.

Techniques. First, restrict to the category $\mathbf{Graph}_{fm} \cup \mathbf{Graph}$ of graphs and full monomorphisms. Then construct a functor $A\colon \mathbf{Graph}_{fm}^{\mathrm{op}} \to \mathbf{CDGA}$ with notation $\mathcal{G} \mapsto A_{\mathcal{G}}$.

Results

Theorem (C-Viruel)**.** *For $\mathcal{G}$ a graph, $A_{\mathcal{G}}$ is an elliptic algebra (hence Poincaré duality). Let $X_{\mathcal{G}}$ be the rational elliptic 1-connected space whose Sullivan minimal model is $A_{\mathcal{G}}$. Then $[X_{\mathcal{G}}, X_{\mathcal{G}}] = \{f_0, f_1\} \cup \mathrm{Aut}(\mathcal{G})$.*

Theorem (C-Viruel)**.** *Every finite group G is realized by infinitely many (non homotopically equivalent) rational elliptic spaces X. That is $G \cong \mathcal{E}(X)$. Moreover, X can be chosen to be the rationalization of an inflexible manifold.*

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Realizing actions

Now what if the group is acting on a $\mathbb{Z}$-module M? Can we realize actions?

The algebraic structure is now (G, M) where G is a group and M is a finitely generated $\mathbb{Z}G$-module. The homotopy invariant is $(\mathcal{E}(-), \pi_k(-))$ where π_k is a $\mathbb{Z}\mathcal{E}(-)$-module.

Our extended problem: is there a finite Postnikov piece X such that the $\mathbb{Z}G$-module M is isomorphic to the $\mathbb{Z}\mathcal{E}(X)$-module $\pi_k(X)$ for some $k \geq 2$.

This is a "dual" of the Steenrod problem. We do not ask for a G-space X but $G \cong \mathcal{E}(X)$. We require X to be a Postnikov piece. If $X = K(M, k)$, then $G = \mathcal{E}(X) \cong \mathrm{Aut}(M)$. We ask $\mathcal{E}(X)$ to act trivially on π_i for $i \neq k$.

This is a harder problem that implies the realizability of groups.

Techniques. The idea is to introduce invariant theory into the picture. If G acts on V then G acts on $\mathbb{Q}[V]$, the ring of polynomial functions, by conjugation. A G-invariant function is a fixed point for this action.

Some results of Hilbert-Noether: if G is finite and V is a faithful $\mathbb{Q}G$-module, then the field of fractions $\mathbb{Q}(V)$ is a Galois extension of $\mathbb{Q}(V)^G$.

Corollary (characterization of finite $G \subset GL(V)$)**.** *There exists $p_1, \ldots, p_r \in \mathbb{Q}[V]$ such that for $f \in GL(V)$, $f \in G$ if and only if $fp_i = p_i$ for all i.*

We modify these algebraic forms:

Lemma. *There exists algebraic forms $q_0, \ldots, q_r \in \mathbb{Q}[V]^G$ satisfying conditions.*

Fix an integer n such that $\deg(q_r) < 2n + 1$ and define a Sullivan minimal model $\mathcal{M}_n$.

Theorem (C-Viruel)**.** *Let G be a finite group, V a finitely generated faithful $\mathbb{Q}G$-module. Then there exist infinitely many (non homotopically equivalent) Postnikov pieces X that realize the action.*

Note: there are three joint papers C-Viruel, a preprint and two in press, that have more details.

Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA 02138
E-mail address: `eriehl@math.harvard.edu`