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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Leanne Merrill **Email/Phone:** leannem@uoregon.edu/518 461 7614

Speaker's Name: William Dwyer

Talk Title: Introduction to operads

Date: 1/27/14 **Time:** 9:30 **am / pm (circle one)**

List 6-12 key words for the talk: ~~[illegible]~~ Little n-disk operad, Lie operad, Koszul duality, Étale homotopy group, Grothendieck-Teichmüller group, Galois theory.

Please summarize the lecture in 5 or fewer sentances: The talk presented two equivalent definitions of operads. There were four main examples of operads given. Then the talk explored uses of operads in homotopy theory and manifold theory. Finally we used operads in algebra and Galois theory, and stated two conjectures in these areas.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
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- ☑ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

William Dwyer - Introduction to Operads 1/27/14

Operads - Tools for describing algebraic structures
- Mathematical objects in their own right

Outline:
1. Operads and their actions on things
2. A particular family of four operads
3. Where operads show up:
   - Homotopy theory
   - Manifold theory
   - Algebra
4. Mash-up.

1 **Definition 1**: monoid $M \xleftrightarrow{\text{modeled on}} \mathrm{Map}(X, X)$

$M \times M \to M$ $\qquad$ $(f, g) \mapsto f(g)$

associative
unital

operad $P \xleftrightarrow{\text{modeled on}} \{\mathrm{Map}(X^r, X),\ r \geq 0\}$

Space $P(r)$, $r \geq 0$ ← arity

$P(r) \times P(l_1) \times \dots \times P(l_r) \longmapsto P(l_1 + \dots + l_r)$ $\qquad$ $(f, g_1, \dots, g_r) \longmapsto f(g_1, \dots, g_r)$

Σ_r acts on $P(r)$ $\qquad$ Σ_r acts on $\mathrm{Map}(X^r, X)$

Σ_* unital, assoc., compatible.

A <u>P-action</u> on X (P-alg)

operad map $P \longrightarrow \{ Map(X^r, X),\ r \geq 0 \}$

$$\coprod_{r \geq 0} P(r) \times_{\Sigma_r} X^r \longrightarrow X \qquad (+\text{associative, unital}).$$

<u>Definition 2</u>: A <u>symmetric</u> <u>sequence</u> is

$\{ S(r),\ r \geq 0,\ \Sigma_r \text{ action on } S(r) \}$

There is a monoidal structure on symm. seq.:

$$(S \circ T)(r) = \left(\coprod_{\substack{k, i_1, \dots, i_k \\ i_1 + \dots + i_k = r}} S(k) \times T(i_1) \times \dots \times T(i_k) \right) \Big/ \Sigma\text{-stuff}$$

Think of $\coprod_{r \geq 0} P(r) \times_{\Sigma_r} X^r$ as

$$\frac{\sum p_r X^r}{r!} \qquad \text{(like a Taylor series).}$$

So we define:

An <u>operad</u> P is a monoid for $\circ$ which is

$P \circ P \to P$, associative, unital.

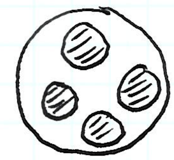
An action of P on a symm. seq. S:

left module $\quad P \circ S \longrightarrow S$

right module $\quad S \circ P \longrightarrow S$

algebra X $\quad P \circ X_{<0>} \longrightarrow X_{<0>}$.

2 Family of four:

Name	Ingredients
- commutative operad $\mathcal{C}$	$\mathcal{C}(r) = *\quad \forall r.$
- associative operad $\mathcal{A}$	$\mathcal{A}(r) = \Sigma_r \quad \forall r.$
- little n-disk operad D_n	$D_n(r) = \{$  $\}$ operad structure: inside each small disk there are smaller disks; composition is just viewing these as small disks inside the larger disk.
- Lie operad (in spectra) $\mathcal{L}$	$\mathcal{L}(r) \simeq \bigvee_{(r-1)!} S^{1-r}$

All of these are tied to each other in duality and in tensor product.

Koszul duality $\quad P \longleftrightarrow P^\vee$

P acts on things $\qquad$ $P^\vee$ acts on cochains on things

$$\mathcal{A} \longleftrightarrow \mathcal{A}$$
$$D_n \longleftrightarrow D_n$$
$$\mathcal{C} \longleftrightarrow \mathcal{L}$$

B-V Tensor Product $P \boxtimes Q$

$P \boxtimes Q$ = a P-algebra in Q-algebras OR
a Q-algebra in P-algebras

Two algebra structures that "commute" with each other in some sense.

$$D_n \simeq D_1 \boxtimes^h D_1 \boxtimes^h \cdots \boxtimes^h D_1 .$$

$\boxtimes^h$ - homotopy derived tensor product.

$$A \sim \pi_0 D_1$$

$$e \sim \operatorname{colim} D_n = D_\infty .$$

Let's try to turn these operads into power series:

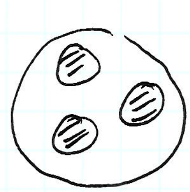
$$C \longleftrightarrow \coprod * \times_{\Sigma_n} X^n \longleftrightarrow \sum \frac{x^n}{n!}$$

$$\mathcal{L} \longleftrightarrow \coprod \bigvee_{(n-1)!} S^{1-n} \wedge_{\Sigma_n} X^{\wedge n} \longleftrightarrow \sum \frac{(-1)^{1-n} x^n}{n}$$

from the Euler characteristic

(3) Where they come up:

3a. Homotopy theory:

point in $D_n(r)$ =  gives $\rightsquigarrow$ $\left(\dfrac{\text{Big disk}}{\partial} \to \dfrac{\text{Big disk}}{\text{complement of small disks}} \right)$

$$= \left(S^n \to \bigvee_r S^n \right).$$

X a pointed space;

$X \wedge S^n \longrightarrow \bigvee_r (X \wedge S^n)$ shows $X \wedge S^n$ is an algebra over $D(n)$ in Top_*^{op}.

X a pointed space:

$\Omega^n X \times \dots \times \Omega^n X \longrightarrow \Omega^n X$ shows $\Omega^n X$ is an algebra over $D(n)$.

Theorem : (Hopkins). n-fold suspensions $\xleftrightarrow{\sim}$ algebras over $D(n)$ in Top_*^{op}.

Theorem : (May, B-V) : n-fold loop space $\xleftrightarrow{\sim}$ algebras over $D(n)$.

3b. Manifold Theory: (Stick to parallelized manifolds)

Suppose N is a framed manifold, n a fixed integer. $n \geq 0$.

Define $N^\Delta(r) = \{ \coprod_r D^n \hookrightarrow N \}$, tangentially straightened.

N^Δ is a right module over D_n.

Theorem (Weiss, Weiss - Boavida de Brito):

If $n = \dim N$, and $\dim M > n+2$, then

$$\mathrm{Emb}^{TS}(N, M) \sim \mathrm{Map}^h_{R+D_n \text{ mod}}(N^\Delta, M^\Delta).$$

(TS = tangentially straightened)

This theorem has computational power!

Theorem (Dwyer - Hess).

$$\left(\begin{array}{c} \text{space of long knots} \\ \text{in } \mathbb{R}^n, n > 3 \end{array} \right) \sim \Omega^2 \mathrm{Map}^h_{\text{operad}}(D_1, D_n)$$

Ω^2 ← double loop space.

$$\left(\begin{array}{c} \text{Vassilier approximation} \\ \text{to long knots in } \mathbb{R}^3 \end{array} \right) \sim \Omega^2 \mathrm{Map}^h_{op}(D_1, D_3)$$

3c. Algebra.

$$F_2 \simeq \pi_1(\mathbb{C}\setminus\{0,1\})$$

$\mathbb{A}^1\setminus\{0,1\}$ defined over $\mathbb{Q}$.

so $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on $\pi_1^{et}(\mathbb{A}^1_{\mathbb{C}}\setminus\{0,1\}) \sim \pi_1(\mathbb{C}\setminus\{0,1\})^\wedge \simeq F_2^\wedge$

et - étale homotopy gp

$\wedge$ - profinitely completed

<u>Grothendieck</u>:

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow GT^\wedge \underset{\text{equations}}{\hookrightarrow} \mathrm{Aut}(F_2^\wedge)$$

<u>Drinfeld</u>:

$$GT_{\mathbb{Q}} \underset{\text{same equations}}{\hookrightarrow} \mathrm{Aut}(F_{2,\mathbb{Q}})$$

<u>Fresse</u>:

$$GT_{\mathbb{Q}} \xrightarrow{\cong} \pi_0 \mathrm{Aut}^h(D_{2,\mathbb{Q}})$$

where GT is the Grothendieck-Teichmüller group.

<u>Question</u>: $GT^\wedge \xrightarrow{\cong} \pi_0 \mathrm{Aut}^h(D_2^\wedge)$?

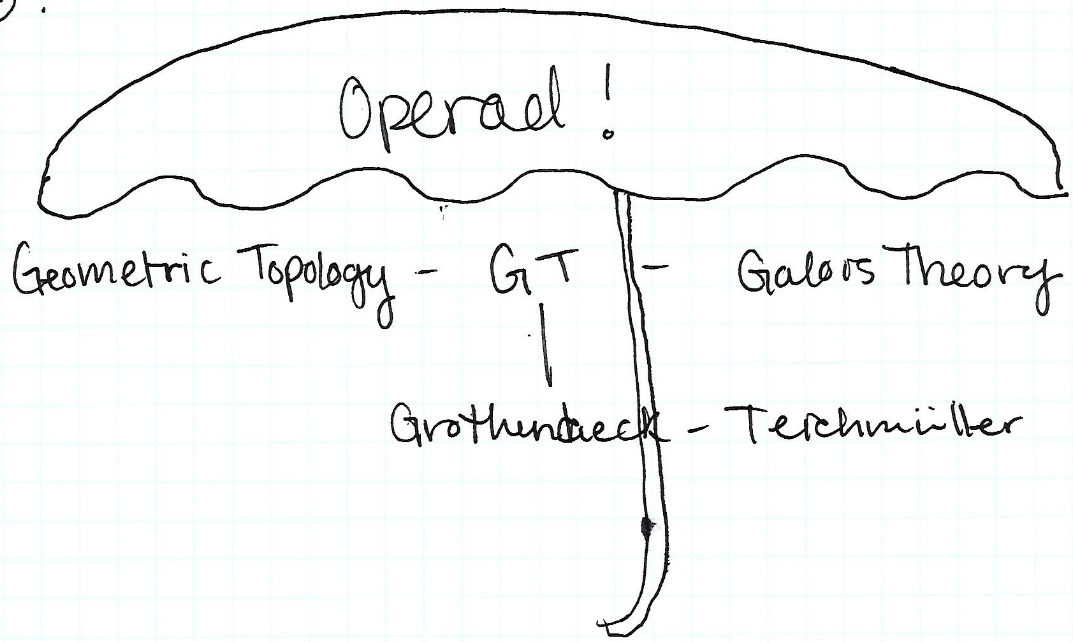
a profinite completion of D_2.

<u>Question</u>: (Dwyer-Hess) $\mathrm{Top}(n) \xrightarrow{\simeq} \mathrm{Aut}^h(D_n)$?

group of automorphisms of $\mathbb{R}^n$.

evidence for second conjecture: $\mathrm{Diff}(D^n, \partial) \simeq \Omega^n \mathrm{Top}(n)/O(n)$.

4.


Operad!
Geometric Topology - GT - Galois Theory
Grothendieck - Teichmüller