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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Leanne Merrill **Email/Phone:** leannem@uoregon.edu / 518 461 7614

Speaker's Name: Michael Ching

Talk Title: Goodwillie's Calculus of Functors

Date: 1 / 27 / 14 **Time:** 11 : 00 **am / pm (circle one)**

List 6-12 key words for the talk: Homotopy functors, polynomial functors, n-excisive functors, manifold calculus, derivatives of functors, Taylor Tower

Please summarize the lecture in 5 or fewer sentances: The talk gave examples of linear functors (reduced homology, stable homotopy, immersion functor). It introduced the notion of 1-excisive functors and introduced first derivatives of general 1-excisive functors. It then generalized the notion of higher excision and convergence of Taylor towers.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
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- **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
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☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

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Michael Ching - Goodwillie's Calculus of Functors 1/27/14

Three Themes: what is calculus?

A study of polynomial approximations.

- What are polynomial functors?
- What do we mean by approximation?
- Classification of polynomials.

A. Motivation and Examples. (Linear functors).

ex 1. $\tilde{H}_* : \underset{\text{based spaces}}{Top_*} \longrightarrow \underset{\text{graded abelian groups}}{gAb}$

why could this be called linear?

$$\tilde{H}_*(X \vee Y) \cong \tilde{H}_*(X) \oplus \tilde{H}_*(Y).$$

More generally $\tilde{H}_*$ has a Mayer-Vietoris sequence:

$$\cdots \to \tilde{H}_*(U \cap V) \to \tilde{H}_*(U) \oplus \tilde{H}_*(V) \to \tilde{H}_*(U \cup V) \to \cdots$$

⌜ Analog: $f : \mathbb{R} \to \mathbb{R}$ is linear if

$$f(x+y) - f(x) - f(y) + f(0) = 0$$ ⌟

What is homology an approximation to?

You can think of homology $\tilde{H}_*$ as a linear approx. to π_*.

There is Hurewicz:

$$\pi_*(X) \xrightarrow{h} \tilde{H}_*(X)$$

This map is an isomorphism under certain circumstances.

Thm If X is k-connected, h is an iso. for $* \leq k+1$
surjection for $* = k+2$.

<u>ex</u> ② Stable homotopy $\pi_*^s : Top_* \to gAb$

- π_*^s is a generalized homology theory.
  and has M-V sequence
- $\pi_*(X) \xrightarrow{s} \pi_*^s(X)$

<u>Frendenthal Theorem</u>: If X is K-connected, s is an iso. for $* \le 2K$, and a surjection for $* = 2K+1$.

So if stable homotopy is the linear approx., we want better (higher degree) approximations.

<u>ex</u> ③ "Manifold/Embedding Calculus" (Weiss)

$$\underset{\text{immersion functor}}{Imm(-,\mathbb{R}^n)} : \underset{\text{open subsets of } \mathbb{R}^m}{\mathcal{O}(\mathbb{R}^m)^{op}} \longrightarrow Top_* .$$

This in also "linear."

$$Imm(U \cup V, \mathbb{R}^n) \simeq Imm(U,\mathbb{R}^n) \times_{Imm(U\cap V,\mathbb{R}^n)} Imm(V,\mathbb{R}^n)$$

(You can also think of this as a sheaf condition.)

Immersions are linear approx. to embedding.

<u>B</u>. Calculus of Homotopy Functors.

(Based on papers by Goodwillie: Calculus I, II, III).

Focus on functors

$$F : Top_* \longrightarrow Top_* . \quad (\text{or } Top_* \to Sp,\ Sp \to Sp, \text{ etc}).$$

(In fact this will encompass earlier examples)

We will insist that functors preserve homotopy equivalences. These will be called <u>homotopy functors</u>.

<u>Definition</u>: A functor F is <u>1-excisive</u> if for any homotopy pushout,

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_2 \arrow[d] \\ X_1 \arrow[r] & X_{12} \end{tikzcd}$$

the square

$$\begin{tikzcd} F(X_0) \arrow[r] \arrow[d] & F(X_2) \arrow[d] \\ F(X_1) \arrow[r] & F(X_{12}) \end{tikzcd}$$

is a homotopy pullback square.

(can think "affine linear" instead of "1-excisive.")

<u>Remark</u>: If F is 1-excisive, then $\pi_* F$ has M-V sequences:

$$\cdots \to \pi_* F(U \cap V) \to \pi_* F(U) \oplus \pi_* F(V) \to \pi_* F(U \cup V) \to \cdots$$

why?
$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & U \cup V \end{tikzcd}$$
is a homotopy pushout provided correct openness conditions on U, V.

<u>Ex</u> $\tilde{H}_*(X) \cong \pi_* \Omega^\infty (H\mathbb{Z} \wedge X)$

$\Omega^\infty(H\mathbb{Z} \wedge -) : Top_* \to Top_*$ is 1-excisive.

Linear Approximation:

Definition: Given a homotopy ~~functor~~ functor $F : Top_* \to Top_*$ and $X \in Top_*$,

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] & CX \arrow[d] & \text{cone} \arrow[l] \\ CX \arrow[r] & SX & \text{unreduced suspension} \arrow[l] \end{tikzcd}$$

is a homotopy pushout.

Then

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(CX) \arrow[d] \\ F(CX) \arrow[r] & F(SX) \end{tikzcd}$$

is a homotopy pullback if F is 1-excisive.

ie: If F is 1-excisive,

$$F(X) \xrightarrow{\sim} F(CX) \times^h_{F(SX)} F(CX) .$$

Let

$$T_1 F(X) := F(CX) \times^h_{F(SX)} F(CX) .$$

Main idea of Goodwillie: $T_1 F$ is "closer" to being linear than F is.

Definition: $P_1 F(X) := \operatorname{hocolim} (F(X) \to T_1 F(X) \to T_1(T_1 F)(X) \to \dots)$

Theorem (Goodwillie): $P_1 F$ is 1-excisive.

If F is 1-excisive, $F \simeq P_1 F$.

In general, the map $F \to P_1 F$ is universal up to homotopy among maps from F to something 1-excisive.

$$\begin{tikzcd} F \arrow[r] \arrow[d] & G \arrow[d, "\wr"] & \text{1-excisive} \arrow[l, bend right] \\ P_1 F \arrow[r] & P_1 G & \end{tikzcd}$$

$F \to G$ factors (uniquely up to homotopy) via $F \to P_1 F$.

one problem: in regular calculus, we must specify a value we take the linear approximation <u>at</u>.

<u>Note</u>: $F(*) \xrightarrow{\sim} P_1 F(*)$. So in a sense this is approximation <u>at</u> the one-point space.

<u>Example</u>: $Id : Top_* \to Top_*$ is <u>not</u> 1-excisive. (and therefore it has interesting calculus).

$$T_1(Id)(X) \simeq * \times^h_{SX} * \simeq \Omega S X \simeq \Omega \Sigma X$$

if X is well-behaved.

so

$$P_1 Id(X) = \text{hocolim}(X \to \Omega\Sigma X \to \Omega^2 \Sigma^2 X \to \dots)$$

$$= \Omega^\infty \Sigma^\infty X = Q(X)$$

(stable homotopy functor).

so

$$\pi_* Id(X) \to \pi_* Q(X) = \pi_*^s(X)$$

Classification of Linear Functors

<u>Theorem</u> (Goodwillie):

Let $F : Top_* \to Top_*$ a homotopy functor, 1-excisive, reduced ($F(*) = *$), and finitary (preserves filtered hocolims).

Then $F(X) \simeq \Omega^\infty(E \wedge X)$ for a spectrum E.

(This is a version of Brown Representability).

E is the first <u>derivative</u> (at $*$) of F.

C. Higher Degree Polynomials.

⌜Analog: $f: \mathbb{R} \to \mathbb{R}$ is quadratic if

$$f(x+y+z) - f(x+y) - \dots = 0$$ ⌟.

Def: A functor $F: \mathrm{Top}_* \to \mathrm{Top}_*$ is n-excisive if for any strongly homotopy cocartesian $(n+1)$-cube

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] \arrow[dr] & X_{n+1} \arrow[dr, dotted, dash] & \\
X_1 \arrow[dr, "\text{p.o.}"] & X_2 \arrow[d] & \vdots \arrow[d] \\
& X_{12} \cdots \arrow[r] & X_{\{1,\dots,n+1\}}
\end{tikzcd}$$

(each two-dimensional face is a pushout)

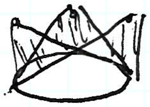
,

$$\begin{tikzcd}
F(X_0) \arrow[r] \arrow[d] \arrow[dr] & F(X_{n+1}) \arrow[dl, dotted, dash] \\
F(X_1) & F(X_2) \\
\vdots \arrow[r] & F(X_{\{1,\dots,n+1\}})
\end{tikzcd}$$

is a homotopy pullback. (just the initial vertex, not each face.)

Def:

$$\begin{tikzcd}
X \arrow[rr] \arrow[dd] \arrow[dr] & & CX \arrow[dd] \arrow[dr] & \\
& CX \arrow[rr] \arrow[dd] & & SX \arrow[ddl] \\
CX \arrow[rr] \arrow[dr] & & SX \arrow[dd] & \\
& SX \arrow[dr] & & \\
& & {} &
\end{tikzcd}$$



(join)

$$F(X) \to \underset{\emptyset \neq U \subseteq \{1,\dots,n+1\}}{\operatorname{holim}} F(U * X) =: T_n F(X)$$

and

$$P_n F(X) = \operatorname{hocolim} (F(X) \to T_n F(X) \to \dots)$$

$F \to \underset{\substack{\uparrow \\ n\text{-excisive} \\ \text{(therefore } n+1 \text{ excisive)}}}{P_n F}$ factors via $F \to P_{n+1} F$.

so we have the Taylor tower of F:

$$F \to \cdots \to P_{n+1} F \longrightarrow P_n F \longrightarrow P_{n-1} F \longrightarrow \cdots \to P_1 F$$

Convergence of Taylor Tower:

For suitable F ("analytic") and X (highly connected), $F(X) \xrightarrow{\sim} \underset{n}{\operatorname{holim}} \; P_n F(X)$.

Ex $F = \mathrm{Id}$. X 1-connected.

Then $X \xrightarrow{\sim} \underset{n}{\operatorname{holim}} \; P_n F(X)$.

why? If X is k-connected, then

$X \to P_n \mathrm{Id}(X)$ is $(n+1)k$-connected.

Other examples of functors to use:

- A functor from K-theory.
- Ω^∞
- Σ^∞
- $\mathrm{Map}(K, X)$