Brooke Shipley - Morita Theory in Stable Homotopy 1/27/14

Morita Theory:

1. Classical Morita theory for Rings
2. Derived
3. DGAs
4. Spectra
5. Recent results

1 Morita Theory for Rings (Morita, 1958)

The following are equivalent.

1. Two rings R and T are Morita equivalent if their module categories are equivalent:
   ie: $\text{mod-}R \cong \text{mod-}T$ (equivalent as categories)
2. $\exists$ a finitely generated projective generator M such that $\hom_T(M, M) \cong R$
3. $\exists$ $R-T$ bimodule N such that
   $$- \otimes_R N : \text{Mod-}R \cong \text{Mod-}T.$$

Sketch of proof:

(2) $\Rightarrow$ (1) $\hom_T(M, -) : \text{Mod-}T \rightleftarrows \text{Mod-}R : - \otimes_R M$

is an equivalence:

Note: $M \longrightarrow R$

$M \longleftarrow R$

Because M is a ~~[illegible]~~ fin. gen. proj. generator, sums and cokernels are preserved, so we can build all modules out of these.

2 $\Rightarrow$ 3 is subsumed above.

3 $\Rightarrow$ 1 is easy.

1 $\Rightarrow$ 2 $M := F(R)$ where F is the hypothesized equivalence.

Note:

$$\hom_T(F(R), F(R)) \cong \hom_R(R, R) \cong R.$$

//

Example: R and $M_n(R)$ are Morita equivalent.

2 Derived:

Ch_R ($\mathbb{Z}$-graded)

$\mathcal{D}(R) := Ch_R(q\text{-iso}^{-1}) \cong Ho(Ch_R)$

This is a triangulated category. (Δ'd)

<u>Derived Morita theory</u> (Rickard '89, '91; Keller '94)

1 Two rings R and T are <u>derived</u> <u>equivalent</u> if $\mathcal{D}(R) \cong_\Delta \mathcal{D}(T)$ equivalent as Δ'd categories.

if and only if

2 $\exists$ compact generator M in $\mathcal{D}(T)$ such that

$$\mathcal{D}(T)(M, M)_* \cong R \quad \text{(concentrated in degree 0)}.$$

M is called a <u>tilting</u> <u>complex</u>.

For $\mathcal{C}$ a Δ'd category with infinite coproducts, M is <u>compact</u> if $\mathcal{C}(M, -)$ preserves sums.

In $\mathcal{D}(R)$, compact $\Leftrightarrow$ q-iso to bounded complex of finitely generated projectives

M is a <u>generator</u> if the only Δ'd subcategory of $\mathcal{C}$ containing M and closed under coproducts is $\mathcal{C}$.

In the theorem above, there is an analog to the bimodule condition in the classical case.

<u>Sketch of proof</u>: "same" as classical case.

1 $\Rightarrow$ 2 $\quad M ::= F(R)$

2 $\Rightarrow$ 1 $\quad \mathcal{D}(T)(M, -) : \mathcal{D}(T) \cong \mathcal{D}(R)$

//

<u>Example</u> (Derived equivalent but <u>NOT</u> Morita equivalent.)

K a field, $M_3(K)$.

$$T = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \quad * \in K. \qquad R = \begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix} \quad * \in K$$

R and T are derived equivalent:

Need to find a tilting complex, M:

<u>Projectives in T</u>: $e_{ii} T = P_i$

$$P_3 = (0\ 0\ *) \hookrightarrow P_2 = (0 * *) \hookrightarrow P_1 = (* * *)$$

Let $M = P_1 \oplus P_2 \oplus P_2/P_3$. M is a tilting complex.

Check: $\mathcal{D}_T(M, M) \cong R$.

But R and T are <u>not</u> Morita equivalent:

Check the indecomposables.

$$T : \begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

$$R : \begin{tikzcd} & \bullet \arrow[dl] \arrow[dr] & \\ \bullet & & \bullet \end{tikzcd}$$

} not the same.

More examples: look at Broué's conjecture.

May lose too much information by passing to derived setting. Perhaps we want to look at the homotopy theories instead. But, actually:

<u>Theorem</u> For rings R and T,

$$D(R) \cong_\Delta D(T) \text{ if and only if } Ch_R \simeq_Q Ch_T$$

(Quillen equivalent)

The above theorem is <u>not</u> true for spectra.

3 <u>Morita Theory for DGAs.</u>

<u>Question</u>: For two DGAs A and B, are the following equivalent?

1. $D(A) \cong_\Delta D(B)$
2. d.g mod - A $\simeq_Q$ d.g. mod - B
3. $\exists\, M$ a compact generator in d.g-mod-A such that $\underline{\hom_A}(M,M) \underset{q\text{-iso}}{\simeq} B$

derived isomorphism complex.

<u>Answer</u>: Start with 3.

3 $\Rightarrow$ 2: $\hom_A(M,-)$ induces a Quillen equivalence.

2 $\rightarrow$ 1: straightforward.

Other directions are false!

1 $\not\Rightarrow$ 2: $\exists$ DGAs A,B s.t. $D(A) \cong_\Delta D(B)$ but $K_*(A) \not\cong K_*(B)$. Thus 2 does not hold.

This was shown by (Schlichting '02, Dugger-Shipley '09).

2 $\not\Rightarrow$ 3 : q-iso's of DGAs in 3 is not enough.
instead need equivalences of spectra.

Fact: Can model DGAs as $H\mathbb{Z}$-algebras, so

$$\text{DGAs} \cong_Q H\mathbb{Z}\text{-algebras}.$$

Counterexample for 2 $\Rightarrow$ 3 :

$$H\mathbb{Z} \xrightarrow[L]{} H\mathbb{Z} \wedge_S H\mathbb{Z}/2 \xleftarrow[R]{} H\mathbb{Z}$$

smash over sphere spectrum

Claim: 1) $L \not\cong R$ as $H\mathbb{Z}$-algebras

2) $L \cong R$ as S-algebras.

Read more in Dugger-Shipley, '07
"Topological equivalences of DGAs."

4 Morita Theory for Ring Spectra (Schweck-Shipley '03)

The following are equivalent:

1 Two ring spectra R and T are Morita equivalent if $\text{Mod-}R \simeq_Q \text{Mod-}T$ are Quillen equivalent.

2 $\exists$ compact generator M such that $\underline{\text{hom}}_T(M,M) \simeq R$

3 $\exists$ an R-T bimodule N such that

$$- \wedge_R N : \text{Mod-}R \xrightarrow[\simeq_Q]{} \text{Mod-}T$$

Sketch of proof: "Similar."

Also can be done for spectral categories (ring spectrum with many objects.)

Everything so far is (mostly) in two surveys:

- Schwede '04
- Shipley '07

(5) Related Results.

1) spaces of Morita equivalences

Thm (Toën, '07)

For two DGAs A, B (or DG-categories)

$$\mathrm{map}^h_{DGAs}(A, B) \simeq N(\text{sp. } A\text{-}B \text{ bimodules}).$$

sp. = "special", meaning quasi-iso to free on one generator as a right B-module.

(Exercise: show $A \to \mathrm{End}(M, M) \simeq B$).

Thm (Dwyer-Hess)

For $(\mathcal{C}, \otimes)$ a monoidal model category satisfying certain axioms, R, T monoids in $\mathcal{C}$, then

$$\mathrm{map}^h_{\text{monoids}}(R, T) \simeq N(\text{sp. } R\text{-}T \text{ bimodules})$$

2) Blumber, Gepner, Tabuada '13:

study spectral categories up to Morita equivalences.

3) Brauer Group:

Def: A an R-algebra is an Azumaya algebra if there exists B such that $A \otimes_R B$ and $B \otimes_R A$ are Morita equivalent to R.

<u>Def</u> Br(R), the Brauer group of R, is the set of Morita equivalence classes of Azumaya algebras over R.

There are derived versions: Toën, Baker-Richter-Szymik

(for commutative ring spectra)

Antieau-Gepner

(for Brauer spectrum)

$\rightsquigarrow$

<u>Thm</u> $\mathrm{Br}\left(\begin{smallmatrix}\text{sphere}\\ \text{spectrum}\end{smallmatrix}\right) = 0$.