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NOTETAKER CHECKLIST FORM

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Name: Leanne Merrill **Email/Phone:** leannem@uoregon.edu/ 518 461 7614

Speaker's Name: Andrew Blumberg

Talk Title: Higher categories and algebraic K-theories

Date: 1/28/14 **Time:** 11:00 am / pm (circle one)

List 6-12 key words for the talk: Waldhausen categories, Algebraic K-theory, Additivity theorem, spectral categories, S. functor, virtual Waldhausen categories.

Please summarize the lecture in 5 or fewer sentances: The lecture introduced traditional K-theory and summarized its basic properties. It defined "up to homotopy" versions of these notions and generalized these in the context of Waldhausen ∞-categories and small spectral categories. It discussed the importance and consequences of the additivity theorem.

CHECK LIST

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Andrew Blumberg: <u>Higher categories and Algebraic K-theory</u>

1/28/14

<u>K-theory</u>: A functor

$$\left\{\begin{array}{c}\text{homotopical category of} \\ \text{(certain) homotopical} \\ \text{categories}\end{array}\right\} \longrightarrow \text{Spectra}$$

Waldhausen:

C a category with distinguish subcategories:

wC - weak equivalences $\xrightarrow{\simeq}$

$\mathrm{cof}\, C$ - cofibrations. $\rightarrowtail$

(C is small, pointed).

These must satisfy certain axioms:

0) isos are in $\mathrm{cof}\, C$, wC.

1) $* \rightarrowtail X$ a cofibration

2)
$$\begin{tikzcd} X \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d] \\ Z \arrow[r, tail] & Z \coprod_X Y \end{tikzcd}$$

3)
$$\begin{tikzcd} Z \arrow[d, "\simeq"'] & X \arrow[l, tail] \arrow[r] \arrow[d, "\simeq"'] & Y \arrow[d, "\simeq"] \\ Z' & X' \arrow[l, tail] \arrow[r] & Y' \end{tikzcd} \implies Z \coprod_X Y \xrightarrow{\simeq} Z' \coprod_{X'} Y'.$$

Have: C, weak equivalences, $\mathrm{Ho}\, C$.

This gives a distinguished collection of homotopy pushouts.

The pushout.

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d] & Y \arrow[d, two heads] \\ * \arrow[r, tail] & Y/X \end{tikzcd}$$

specifies which maps have homotopy cofibers.

and the pushout

$$\begin{tikzcd} * \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d] \\ X \arrow[r] & X \amalg Y \end{tikzcd}$$

is an exact sequence.

<u>S.C.</u>

$S_n C$ $\quad Ar[n] \longrightarrow C$

$(i,j) \quad 0 \le i \le j \le n$

$(i,j) \longrightarrow (i',j') \quad i \le i', \ j \le j'$.

Conditions:

- $A_{ii} = * \quad \forall i$
- $A_{ij} \rightarrowtail A_{ik} \quad i,j \le k$.
- $$\begin{tikzcd} A_{ij} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & A_{ik} \arrow[d] \\ * = A_{jj} \arrow[r] & A_{jk} \end{tikzcd}$$ a ~~...~~ cofibration.

S.C. has a simplicial structure.

It is a (Waldhausen) category.

Can view as $\Omega | N^w S.C. |$.

From Blumberg-Mandell:

<u>Definition</u>. $X \to Y$ is a weak cofibration if $\overset{\simeq}{\rightsquigarrow}$ $X' \rightarrowtail Y'$ a cofibration.

A homotopy cocartesian square:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ W \arrow[r] & Z \end{tikzcd} \quad \overset{\simeq}{\longleftrightarrow} \quad \begin{tikzcd} X' \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y' \arrow[d] \\ W' \arrow[r] & Z' \end{tikzcd}$$

$S'_\cdot C$

$$* \overset{\simeq}{\rightarrowtail} A_{ii}$$

$A_{ij} \longrightarrow A_{ij}$ weak cofibration.

$$\begin{tikzcd} & A_{ij} \arrow[r] \arrow[d] & A_{ik} \arrow[d] \\ * \arrow[r, tail, "\simeq"] & A_{jj} \arrow[r] & A_{jk} \end{tikzcd}$$

homotopy cocartesian

The $S'_\cdot C$, version of K-theory is functorial weakly exact functors

→ preserve weak equivalences

→ preserve homotopy cocartesian.

Algebraic K-theory is invariant of "underlying homotopy theory of C."

Given $C \xrightarrow{F} D$, when is $K(F) : K(C) \xrightarrow[\simeq]{} K(D)$?

Possible answers:

1) if $\mathrm{Ho}\, C \to \mathrm{Ho}\, D$ is equivalence?

Thomas-Trobaugh show this is true in the presence of <u>stability</u>.

(DG - Waldhausen categories).

2) What if $LF: LC \to LD$ is a Dwyer-Khan equivalence of simplicial categories?

That is, C - a model catecory, then

$$LC(X,Y) \simeq C(X^{cf}, Y^{cf}).$$

Toën-Vezzosi show: if $LC \simeq LD$, then $K(C) \simeq K(D)$.

Cisinski, Blumberg-Mandell show

$$\begin{array}{ccc} S_n C & \leftrightarrow & LC(-,-) \times LC(-,-) \times \cdots \\ \downarrow & & \\ \coprod B\mathrm{haut} & & \end{array}$$

coproduct of homotopy automorphisms of objects

This is to ~~also~~ say that:

Algebraic K-theory is invariant of underlying ∞-categories of C.

$N^{hc}((LC)^{fib})$ - $\left\{ \begin{array}{l} \text{functorial at} \\ \text{this level.} \end{array} \right.$

The point is to set up the following questions:

1) What kind of functor is K-theory?

2) Can we build K-theory directly from ∞-categorical data?

Slogan: Algebraic K-theory <u>splits</u> exact sequences.

$K_0(C) :=$ free abelian group on $[M]$

$$X \rightarrowtail Y \to Y/X$$

$$[X] + [Y/X] = [Y]$$

That is, we can't tell the difference between

$$\left.\begin{array}{l} X \rightarrowtail Y \to Y/X \\ X \rightarrowtail X \amalg Y/X \twoheadrightarrow Y/X \end{array}\right\} \text{ same in } K_0.$$

<u>Additivity Theorem</u>: C a Waldhausen category.

$S_2 C$:

$$\begin{tikzcd} * \arrow[r, tail] & A_{01} \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & A_{02} \arrow[d] \\ & * \arrow[r, tail] & A_{12} \end{tikzcd}$$

category of exact sequences in C.

Waldhausen proved: $K(S_2 C) \simeq K(C) \times K(C)$

The maps are:

$$A \rightarrowtail B \to B/A \longmapsto (A, B/A)$$

$$A \rightarrowtail A \amalg B/A \to B/A \longleftarrow (A, B/A)$$

Waldhausen derives everything else we know about K-theory from the additivity theorem.

Staffeldt (based on Greyson) prove most of Quillen's theorems from additivity thm.

McCarthy proved additivity thm via an explicit simplicial homotopy.

These arguments work very broadly (Hesselholt-Madsen).

Suppose F is a functor:

$$F : \begin{matrix}\text{(small)}\\ \text{categories}\\ \text{with cofibrations}\end{matrix} \longrightarrow \text{Spaces}$$

so that $F(*) = *$.

$F(C \times D) \simeq F(C) \times F(D)$.

and if $F(C_n) \xrightarrow{\simeq} F(D_n)$ $\forall n$,

then $|F(C_\bullet)| \longrightarrow |F(D_\bullet)|$.

Then $|F(S_\bullet -)|$ has the additivity theorem.

That is: $S_\bullet$ is the universal way to force a functor to be additive.

Want: to characterize algebraic K-theory in terms of additive functors on ∞-categories.

there are two approaches to this:

1) Borwick: start with "Waldhausen ∞-categories"

2) Tabuada, Blumberg-Gepner-Tabuada

$$\mathrm{id} \rightarrowtail C \longrightarrow \Sigma \quad \text{exact}.$$

Then $K(\mathrm{id}) \vee K(\Sigma) \simeq *$, so

$$K(\Sigma) \simeq -K(\mathrm{id}).$$

Continuing with Tabuada's approach:

start with small stable ∞-categories (idempotent complete).

or: start with small spectral categories:

then: $C \xrightarrow[\simeq]{} \mathrm{Fun}(C^{op}, \text{spectral})^{\omega}$

How do we see additivity data?

Thomason-Trobaugh: If $A \to B \to C$ stable Wald. cats. such that

$H_0 A \to H_0 B \to H_0 C$ and

$H_0 C \simeq H_0 B / H_0 B$,

(Verdier quotient)

then $K(A) \to K(B) \to K(C)$.

This says that on data on the level of triangulated categories, can see the K-theory information we want.

suppose instead we ask the following question:

we require:

$$\begin{tikzcd} A \arrow[r, "f"'] & B \arrow[l, bend right, "h"'] \arrow[r, "g"'] & C \arrow[l, bend right, "i"'] \end{tikzcd}$$

and "$C \to S_2 C \to C$" is a sequence of this kind

$h \circ f \simeq id$, $g \circ i \simeq id$

and that we preserve filtered colimits of module categories.

Then:

Borwick	**Blumberg – Gepner – Tabuada**
$\mathrm{Exc}\left(\mathrm{Fun}(\mathrm{Wald}_\infty^{op,w}, \text{spaces})/\sim\right)$	$\mathrm{Stab}\left(\mathrm{Fun}((\mathrm{Cat}^{ex}_{\infty,st}, st)^{w,op}; \text{spaces})\right)/\sim$
$\mathcal{V}$ "virtual waldhausen"	$\mathcal{M}$

so $\mathrm{Fun}^L(\mathcal{M}, \text{Spectra}) \simeq \mathrm{Fun}_{\substack{add \\ FH}}(\mathrm{Cat}^{ex}_{\infty,st}, \text{spectra})$

Consequences:

1) K-theory is corepresentable in $\mathcal{M}$.

$$\mathrm{Map}_{\mathcal{M}}\left(\begin{smallmatrix}\text{corepresentable}\\ \text{spectra}\end{smallmatrix}, A\right) \simeq K(A).$$

Yoneda lemma lets us study trace maps.

2) In $\mathcal{M}$, $S_\bullet$ is suspension on the nose.

Can view K-theory as a Goodwillie derivative

3) In $\mathcal{V}$, one gets new proofs of theorems of Waldhausen (see Fiore – Lück).

New questions / things to consider:

1) Multiplicative structure (see Elmendorf-Mandell, Blumberg-Mandell, Barwick, Blumberg-Gepner-Tabuada, Glasman, Gepner-Groth-Nikolaus)

2) What about TC, THH? Is there an analogous characterization? (Related to a conjecture of Kaledin)

What is the conceptual explanation for
$$TC(S)^\wedge_p \simeq S \vee \Sigma \mathbb{CP}^\infty_{-1} \quad ?$$

All of this provides a conceptual framework for things we have known about K-theory for a long time. Can this perspective provide anything new?

ALGEBRAIC K-THEORY AND HIGHER CATEGORIES

ANDREW J. BLUMBERG

Abstract. The outline of the talk.

1. Setup

- **Goal: Explain algebraic K-theory as a functor from the homotopical category of homotopical categories to spectra.**
- Start with a "classical" model of a homotopical category.

  Definition 1.1. A pointed category $\mathcal{C}$ equipped with subcategories $\mathrm{cof}(\mathcal{C})$ of cofibrations and $w\mathcal{C}$ of weak equivalences is a Waldhausen category if:
  (1) Every isomorphism is a cofibration.
  (2) The unique map $* \to X$ is a cofibration for every X.
  (3) If $X \to Y$ is a cofibration and $X \to Z$ is any map, the pushout

  $$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & Y \coprod_X Z \end{tikzcd}$$

  exists and $Z \to Y \coprod_X Z$ is a cofibration.
  (4) Every isomorphism is a weak equivalence.
  (5) Given a diagram

  $$\begin{tikzcd} Y \arrow[d] & X \arrow[l] \arrow[r] \arrow[d] & Z \arrow[d] \\ Y' & X' \arrow[l] \arrow[r] & Z' \end{tikzcd}$$

  where $X \to Y$ and $X' \to Y'$ are cofibrations and the vertical maps are weak equivalences, the induced map $Y \coprod_X Z \to Y' \coprod_{X'} Z'$ is a weak equivalence.

- What is this data? This is a category with weak equivalences and certain specified homotopy pushouts. In particular, we are stipulating which maps have homotopy cofibers:

  $$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ * \arrow[r] & Y/X. \end{tikzcd}$$

  (This is an exact sequence.)

- Also, we have coproducts (and this becomes a symmetric monoidal category under coproduct).
- Out of this data, we can define the Waldhausen K-theory space:

  Definition 1.2. For each n, the simplicial space $S_\bullet\mathcal{C}$ is specified as functors
  $$\mathrm{Ar}([n]) \longrightarrow \mathcal{C}$$
  (i.e., where $\mathrm{Ar}([n])$ has objects (i,j) with $0 \leq i, j \leq n$, and maps if $i \leq i'$ and $j \leq j'$) with certain properties:

  Specifically, a collection of objects A_{ij} such that
  (1) $A_{ii} = *$ for all i.
  (2) $A_{ij} \to A_{ik}$ is a cofibration for all i, j, and k.
  (3) Each square
  $$\begin{tikzcd} A_{ij} \arrow[r] \arrow[d] & A_{ik} \arrow[d] \\ * \arrow[r] & A_{jk}. \end{tikzcd}$$

  (This is itself a Waldhausen category, of course.)
- Now, we can relax the hypotheses a bit, as follows. We define a homotopy pushout square to be a square that is equivalent via a zig-zag to a pushout along a cofibration. (Notion of a weak cofibration is useful here.)

  Under mild hypotheses, these behave the way we expect them to.
- Now can redo the $S_\bullet$ construction; call it the $S'_\bullet$ construction. (Joint with Mandell.)
- Functorial in "weakly exact" functors (preserve the point (up to homotopy), weak equivalences and weak cofibrations).
- This sure makes it look like algebraic K-theory reflects the homotopical data encoded by the Waldhausen category structure (i.e., weak equivalences and homotopy pushouts).
- First guess: $F\colon \mathcal{C} \to \mathcal{D}$ induces an equivalence of homotopy categories, then F induces an equivalence of algebraic K-theory.
- Thomason-Trobaugh proved this is true, under stability hypotheses (DG-Waldhausen categories).
- More sophisticated: $F\colon L^H\mathcal{C} \to L^H\mathcal{D}$ is a DK-equivalence of simplicial categories, then F induces an equivalence on algebraic K-theory. Toen-Vezzosi, Cisinski, Blumberg-Mandell. (Equivalent to approximation theorem.)
- In fact, we (Blumberg-Mandell) explain how K-theory is assembled from the mapping spaces in the Dwyer-Kan simplicial localization (total space of fibration with base homotopy automorphism spaces and fiber mapping spaces).
- Interlude about ∞-categories: What this says is that algebraic K-theory is an invariant of the underlying ∞-category. (One way to extract it is from a fibrant replacement of the Dwyer-Kan simplicial localizations; also, Barwick-Kan relative category.)
- More precisely, algebraic K-theory is a functor from ∞-categories with certain homotopy pushouts to the ∞-category of spectra.

- A fruitful question to ask: What kind of functor is it? (And can we build it directly?)

2. Universal characterizations of higher algebraic K-theory

- Slogan: Algebraic K-theory splits exact sequences.
- In K_0, this is the definition:

  Definition 2.1. For a Waldhausen category $\mathcal{C}$, K_0 is the free abelian group on weak equivalence classes $[M]$ subject to the relation
  $$[Y] = [X] + [Y/X].$$
  (Notice that this means that the exact sequence $X \to Y \to Y/X$ and $X \to X \coprod Y/X \to Y/X$ become equivalent.)
- In higher algebraic K-theory, this becomes Waldhausen's additivity theorem. Recall that $S_2\mathcal{C}$ is the category of exact sequences. Additivity theorem says that:
  $$K(S_2\mathcal{C}) \simeq K(\mathcal{C}) \times K(\mathcal{C})$$
  via the obvious map.
- Waldhausen deduces almost all of the rest of his theorems from the addivity theorem. Staffeldt showed that all of Quillen's foundational theorems follow from it too.
- McCarthy gave a marvellous proof of the addivitity theorem:

  Theorem 2.2. *If F is a functor from Waldhausen categories to spectra such that*
  1. *F takes the trivial category to a point,*
  2. *F preserves products up to weak equivalence, and*
  3. *Realization property for geometric realization,*

  then $F(S_\bullet -)$ has the additivity theorem.
- Put another way, the $S_\bullet$ construction is the universal thing that forces a functor to be additive.
- Following McCarthy, Hesselholt-Madsen explain how this basically implies all the rest of Waldhausen's theorems. (Analogue of the Staffeldt observation.)
- So we want to express the idea that K-theory is somehow a distinguished functor that "satisfies additivity".
- Question: what is the domain category?
- We start with the homotopical category of Waldhausen categories.
- One choice: Observe that the additivity theorem implies that K-theory is an invariant of stable categories: under reasonable hypotheses, there is a cofiber sequence
  $$\mathrm{Id} \longrightarrow CX \longrightarrow \Sigma X,$$
  and then we conclude that $\mathrm{Id} \vee \Sigma \simeq *$.
- So we can work with small stable categories; think of this as a Morita-theoretic context. (Model this by taking spectral categories and forcing the map $\mathcal{C} \to Fun(\mathcal{C}^{\mathrm{op}}, S)^{\omega}$ to be an equivalence.
- OK, so we have functors out of some homotopical category of small homotopical categories with colimits (e.g., small stable categories), and we want to identify those functors which have the additivity theorem.

- First, we want to observe that we can reflect the addivity theorem in terms of properties of the the category of Waldhausen categories (or small stable categories):
- Thomason-Trobaugh tell us that given stable categories lifting triangulated categories, if we have a Bousfield localization sequence
$$\mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{B}/\mathcal{A},$$
then we have a cofiber sequence on K-theory. This is a bit stronger than additivity (it's the property we see on non-connective K-theory), but related.
- Additivity can be seen as a property in the same way:
$$\begin{tikzcd} A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "h"] \arrow[r, shift left, "g"] & C \arrow[l, shift left, "i"] \end{tikzcd}$$
where (f, h) and (g, i) are adjoint pairs such that $h \circ f \simeq \mathrm{id}$ and $g \circ i \simeq \mathrm{id}$ (Split Bousfield localization sequences, basically.)
- Here's our strategy: we're going to modify the category of Waldhausen categories so that colimit-preserving functors out of it are the same as additive functors.
- Additional property: preserve filtered colimits.
- Easy as pie:
  Take the compact objects, take simplicial presheaves, localize, stabilize. (Work of Tabuada and Blumberg-Gepner-Tabuada.)
  (In Barwick's setting, take compact objects, take certain simplicial presheaves, localize, take excisive functors.)
- Some interesting consequences:
  (1) BGT: Get a "category of motives"; K-theory is co-representable (initial additive functor over the "moduli of objects"), can view this as "applying K-theory to the hom objects".
  (2) Barwick: new proofs of the theorems of Waldhausen (also, Fiore).
  (3) The $S_\bullet$ construction becomes the suspension; Barwick observes this means K-theory can be thought of as a derivative (in the Goodwillie sense). (Waldhausen and McCarthy knew this, of course.)
  (4) Yoneda lemma gives trace maps.
- Other consequences, future questions:
  (1) Multiplicative structures: Many authors — (Cisinski-Tabuada, Blumberg-Mandell, Blumberg-Gepner-Tabuada, Barwick, Glasman, Gepner-Groth-Nikolaus, Elmendorf-Mandell).
  (2) Trace methods (how to fit TC and THH into this perspective; Blumberg-Mandell work on cyclotomic spectra, Kaledin, etc.)
  (3) Other kinds of K-theory (endomorphisms, homotopy invariant, etc.)
  (4) Even higher categories. (Ayala-Blumberg.)
  (5) Relationship to field theories.
  (6) What can we do with this?

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