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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Leanne Merrill **Email/Phone:** leannem@uoregon.edu/ 5184617614

Speaker's Name: Christopher Douglas

Talk Title: Towards explicit models for higher K-theories

Date: 1/28/14 **Time:** 2:00 am / pm (circle one)

List 6-12 key words for the talk: virtual vector spaces, Fredholm ~~operators~~ operators, Topological field theory, quantum fields, trace class operators, TMF.

Please summarize the lecture in 5 or fewer sentances: The lecture focused on analogies between K-theory and virtual vector spaces, fredholm operators, and semigroups of trace class operators. It explored in detail the one dimensional pictures of topological field theories and used them to build intuition for higher dimensional versions. It ended with a question about the connection between TMF and 2-fredholm quantum fields.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - <u>**Computer Presentations**</u>: Obtain a copy of their presentation
  - <u>**Overhead**</u>: Obtain a copy or use the originals and scan them
  - <u>**Blackboard**</u>: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - <u>**Handouts**</u>: Obtain copies of and scan all handouts
- ☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack <u>with this completed sheet on top</u> and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☑ Email the re-named files to <u>notes@msri.org</u> with the workshop name and your name in the subject line.

Christopher Douglas - Towards an explicit model for higher k-theory ①

1/28/14

Joint work with Henriques.
Context w/ Bartels, Henriques.
Based on Segal, Stolz-Teichner.

Situation:

algebra: cohomology, K-theory, TMF ...

topology: cells, vector bundles, ? ...

Understanding of algebra has outstripped that of topology. How can we extend the topology?

From vector spaces ...

... to k-theory.

consider $\left\{ \begin{matrix} V \\ \downarrow \\ X \end{matrix} \right\} \Big/ \sim$ or, better... $\left\{ \begin{matrix} V \ominus W \\ \downarrow \\ X \end{matrix} \right\} \Big/ \sim$

vector bundle; virtual vector bundle

$K^0(X) := \left\{ \begin{matrix} V \ominus W \\ \downarrow \\ X \end{matrix} \right\} \Big/ \sim$.

for simplicity

for simplicity $X = *$

suggests {"virtual vector spaces"} $\sim$ K.

... to topological field theory.

given V a vector space, have

$$\text{TFT} \quad \mathrm{Bord}_0^1 \longrightarrow \mathrm{Vect}$$
$$\bullet \longmapsto V$$
$$\supset \longmapsto V^* \otimes V \xrightarrow{ev} \mathbb{C}$$

suggests K theory $\rightsquigarrow$ 1-dim TFT.

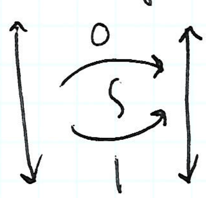
K-Theory via Fredholm operators:

Encode $V \subset H \leftarrow$ a fixed infinite dim. ambient Hilbert space.

as: $(F : H \to H \rightsquigarrow \ker F \subset H)$
$\rightsquigarrow \operatorname{coker} F \subset H$

<u>Def</u>: $F : H \to H$ is <u>Fredholm</u> if $\ker F$, $\operatorname{coker} F$ are finite dimensional.

<u>Note</u>: Topology on operators allows cancelling; eg in the one dimensional case.



$$\Rightarrow \quad \mathbb{R} \ominus \mathbb{R} \text{ is } 0 \ominus 0$$

<u>Model</u>: {Fredholm operators} $\sim K$.

ie, it is a classifying space for K theory.

Map is: $F \longmapsto \ker F \ominus \operatorname{coker} F$.

Interlude: operators as quantum fields.

Infinite dimensional H provides partial TFT:

$$\left(\underset{H}{\bullet}\ ,\ \underset{id_H}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet} \right)$$

operator F provides an extension to $-\!\!\!\times\!\!\!-$ (manifolds with marked points)

this assigns $-\!\!\!\times\!\!\!- \quad H \xrightarrow[F]{} H$

Def A quantum field is

$$\left(\begin{tikzcd} \mathrm{Bord}^1_0 \arrow[r] & \mathrm{Vect} \\ \mathrm{Bord}^{1,m}_0 \arrow[u, phantom, "\cap"] \arrow[ur, dashed] & \end{tikzcd} \right)$$

(m stands for marked pt).

Model: $\left\{ \begin{array}{l} \text{Fredholm quantum fields} \\ \text{for } 1\text{-dim pTFT} \end{array} \right\} \sim K$.

K-Theory via trace class semigroups:

Deform a Fredholm operator F:

dots are eigenvalues.

F :

finite dimensional eigenspaces

possibly infinite dimensional eigenspaces

to:

$\tilde{F}$:

∞

Unbounded Fredholm op.

by pushing infinite dimensional eigenspaces to infinity.

to: $\exp e^{-t\tilde{F}}$

0

trace class operator

giant kernel from ∞ above

Actually have a one-parameter family of trace class operators.

Model# : $\left\{ \begin{matrix} \mathbb{R}_+ \text{ - semigroups} \\ \text{of trace class ops} \end{matrix} \right\} \sim K$

- sweeping things under the rug.

Interlude: trace class semigroups as QFTs.

Deform pTFT $Z_H = \left(\underset{H}{\bullet},\ \underset{id_H}{\bullet\!\!-\!\!-\!\!\bullet} \right)$

using quantum field F

to pQFT Z_H^F (a QFT is like a TFT except that manifolds in question have metrics).

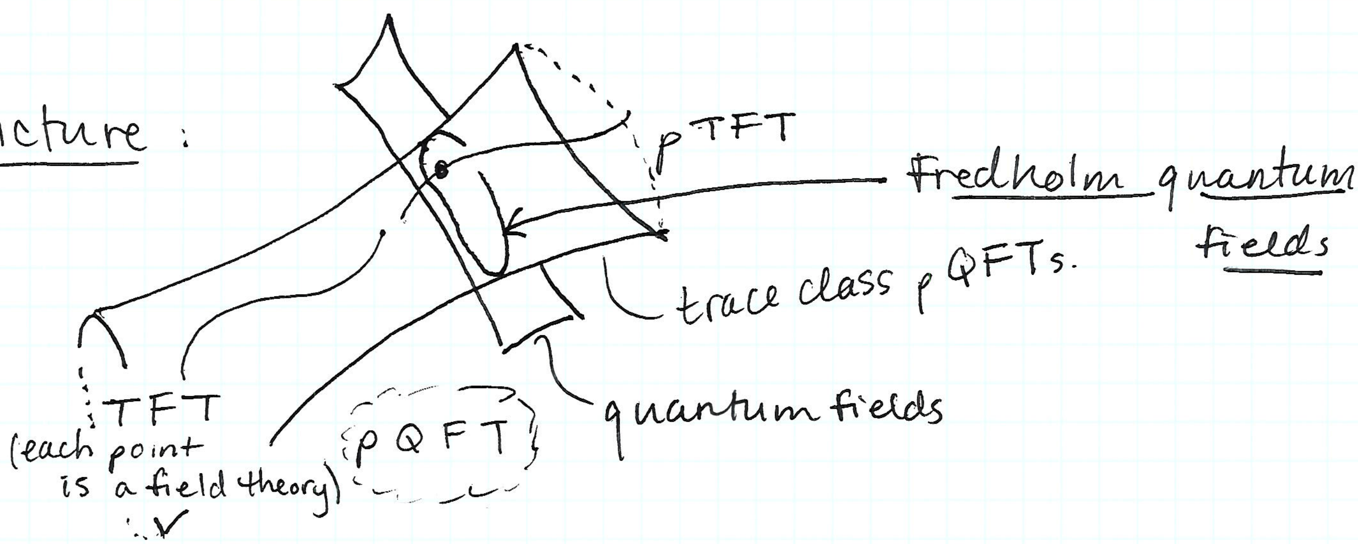
where:

$$Z_H^F\left(\underbrace{\quad\quad}_{t} \right) = \text{"} \bullet\!\times\!\times\!\times\!\times\!\bullet \text{"}$$

$$:= \lim_{n\to\infty} \overbrace{\bullet\!\underbrace{\times}_{1-\frac{t}{n}F}\!\times\!\times\!\times\!\bullet}^{n} = e^{-tF}$$

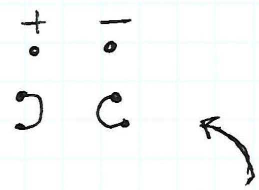
Model# : $\left\{ \begin{matrix} \text{trace class} \\ \text{pQFT} \\ \text{dim. } 1 \end{matrix} \right\} \sim K$.

Picture :



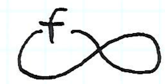
How to go to higher dimensions?
Specifically, what do "trace class" and "Fredholm" mean in two dimensions?

Trace class and traces



Dualizable:

$V \in \mathcal{C}$ dualizable $\rightsquigarrow$ TFT $\mathrm{Bord}_0^1 \to \mathcal{C}$

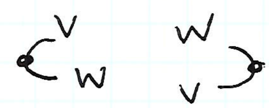
for $f : V \longrightarrow V$, trace $tr(f) =$  $(= Z_V(\eta, f),\ \eta \in \pi_1^s)$.

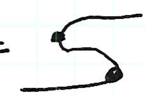
· agrees with notion of trace of a matrix.

Non-dualizable:

$V \in \mathcal{C} \rightsquigarrow$ pTFT. Not clear how to define trace.

<u>Def</u> (Stolz – Teichner):

$f : V \to V$ is <u>trace class</u> if $\exists$ 

$- f - =$ 

<u>Exercise</u>: dualizable admit trace class operators.

<u>Def</u> If f is trace class, then

$tr(f) =$ 

<u>Note</u>: 

So this reproduces the usual notion of trace in the dualizable context.

2-trace class and 2-traces:

Dualizable

$A \in \mathcal{C}^2$ dualizable, $\phi : id_A \Rightarrow id_A$

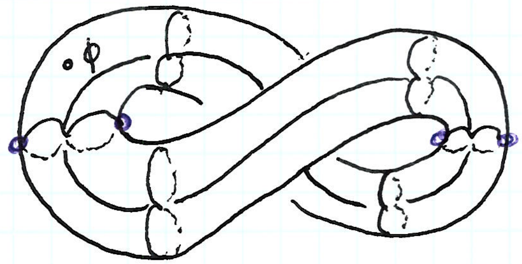
Drawn as:  $\Uparrow$

Then

field theory provided by cobordism hypothesis

$$tr_2(\phi) = Z_A(\eta^2, \phi).$$

Picture :

$\eta^2 =$ 

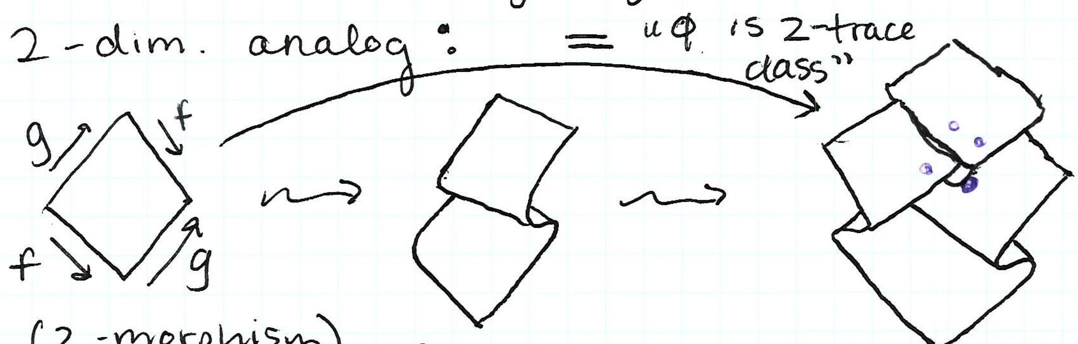
4 morse critical points.

Take a figure-eight, and move it around in $\mathbb{R}^3$ in the shape of a figure-eight.

Non-dualizable :

Theme : field theory beyond the cobordism hypothesis.

2-dim. analog :



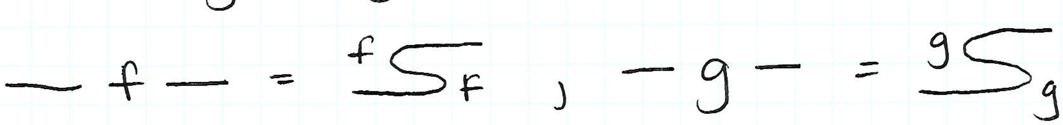
4 Morse critical points (3 are hidden)

(2-morphism)

(fold in one direction)

(fold in other direction)

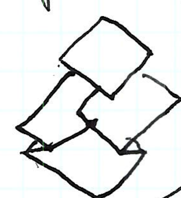
Def If $A \in \mathcal{C}^2$, $f : A \to A$, $g : A \to A$, we say $\phi : fg \Rightarrow gf$ is 2-trace class if



and (insert pictures of board here).

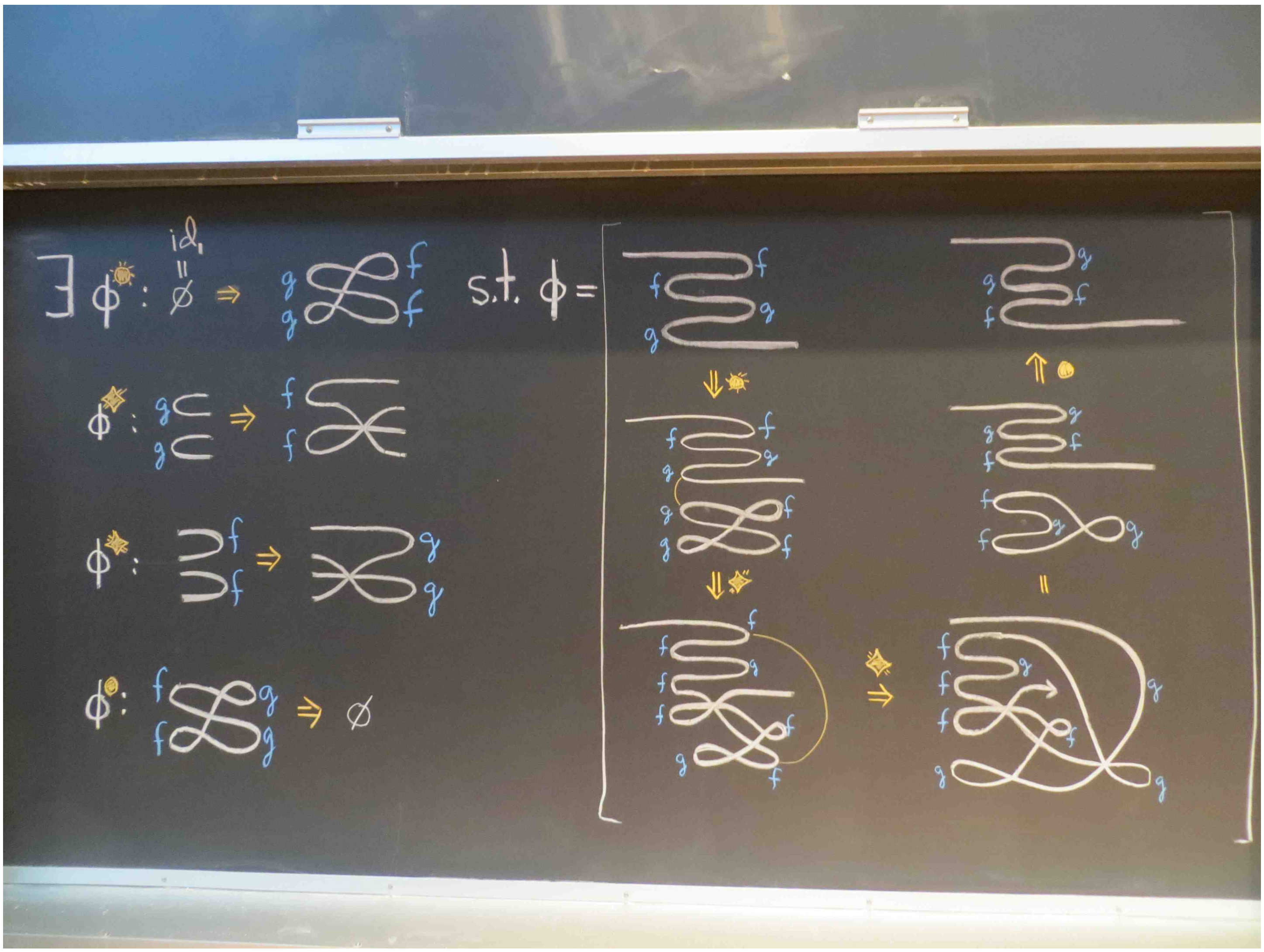
The 1-trace is another name for Hoschild homology.
The 2-trace is another name for $S^1 \times S^1$ Hoschild homology.

Summary:

$\underline{K}$	ingredients (1-dim)	ingredients (2-dim)	TMF
$\wr$	vect. sp. $\sim$ TFT	CFT	$\wr$???
{Fred. ops}	Hilb sp $\sim$ pTFT	pCFT	{2-Fred q. fields}
	operator $\sim$ q field	q field	
	RG		
$\wr$			
{tr. cl. semigroups}	semigroups $\sim$ pQFT_1 $\rightsquigarrow$	pQFT_2	({2-tr. cl. pQFT_2})
	tr. cl. $\sim$ Z $\rightsquigarrow$		

Question:
What evidence is there that this models TMF?

Theorem: QFT_2 gives...
Stolz–Teichner: modular forms
DH: string group
C: Witten genus
BE: TMF $\otimes$ $\mathbb{C}$


∃ ϕ : id = ∅ ⇒
s.t. ϕ =
ϕ :
ϕ :
ϕ : ⇒ ∅

Def: for ϕ 2-trace class, the <u>2-trace</u> is