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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Leanne Merrill **Email/Phone:** Leannem@uoregon.edu/518-4617614

Speaker's Name: Julie Bergner

Talk Title: Models for homotopical higher categories

Date: 1/28/14 **Time:** 3:30 am / pm (circle one)

List 6-12 key words for the talk: ∞-categories, (∞,n)-categories, Cobordism categories, segal spaces, Θ-construction, Θ_n spaces.

Please summarize the lecture in 5 or fewer sentances: This lecture gave heuristic descriptions and definitions of the notions of higher categories. It gave several examples of these categories. it talked about the difficulties of extending naive definitions to higher dimensions, and then used segal spaces and Θ_n spaces to give the correct construction.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
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  - **Handouts**: Obtain copies of and scan all handouts
- ☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☑ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Julie Bergner - Models for homotopical higher categories 1/28/14

Idea of higher categories:

objects

1-morphisms between objects

2-morphisms between 1-morphisms :

Picture:
$$\begin{tikzcd} \bullet \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$

dots: objects
arrows $\longrightarrow$: 1-morphisms :
arrows $\Longrightarrow$: 2-morphisms.

⋮

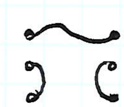
n-morphisms between (n-1)-morphisms
form an "n-category"

⋮

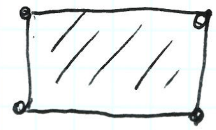
keep going, form an ∞-category.

No problem if everything is strict in the definition.
(associativity, identities, etc., are fine).

But examples "from nature" are often not strict.
eg. associativity only up to isomorphism.
Necessitates coherence conditions.

Ex: Cobordism (higher) categories
objects: 0-manifolds ex: 
1-morphisms: 1-dimensional cobordisms.

This is not associative "on the nose."
Can fix with homotopy.
Now, 2-morphisms: 2-dimensional cobordisms.



But only works for top dimension.
So this does not form a strict 2-category.

There are lots of definitions of weak n-categories, but they are hard to compare.

A more tractable version is the homotopical version.

Definition: An (∞, n)-category is a (weak) ∞-category where k-morphisms are (weakly) invertible for $k > n$. (not a rigorous definition YET.)

Back to cobordism example: there is an (∞, n)-cat. version:

Ex Cobordism (∞, n)-category.

0-manifolds
$\vdots$
n-dimensional cobordisms $\rightsquigarrow$ n-morphisms.
diffeomorphisms $\rightsquigarrow$ $(n+1)$-morphisms
isotopies $\rightsquigarrow$ $(n+2)$-morphisms.
$\vdots$

These are invertible in the weak sense.

How do we make the definition of (∞, n)-category more rigorous?

Definition: An $(\infty, 0)$-category (or ∞-groupoid) is just a topological space / simplicial set.

Why? X - a space. Then

points of X $\rightsquigarrow$ objects

paths of X $\rightsquigarrow$ 1-morphisms

(these are weakly invertible).

homotopies between paths $\rightsquigarrow$ 2-morphisms.

$\vdots$ all morphisms at all levels are invertible.

So this makes sense as a definition.

We have nice homotopy theory for spaces.

$\rightsquigarrow$ model category structure.

Key Property: model structure is cartesian, so internal hom objects are compatible with model structure.

Want: An (∞, n)-category should be a category enriched in $(\infty, n-1)$-categories.

This gives a natural candidate for one model of $(\infty, 1)$-categories.

First approach to $(\infty, 1)$-category:

categories enriched in spaces.

$\mathrm{Map}(x, y)$ is a space.

objects

points are now 1-morphisms.

paths are now 2-morphisms

$\vdots$

for level 2 and beyond, morphisms are weakly invertible.

Simplicial categories: categories enriched in simplicial sets.

we can do homotopy theory for these simplicial categories $\rightsquigarrow$ have model category structure.

we can define categories enriched in simplicial categories as $(\infty, 2)$-categories, but there will be 2 problems:

- too strict for examples
- no longer have a model structure. Why not?

  model category of simplicial categories is not cartesian, so we cannot do homotopy theory.

Luckily, we have other approaches to $(\infty, 1)$-categories.

- quasi-categories (Boardman-Vogt, Joyal, Lurie)
- Segal categories (Dwyer-Kan-Smith, Simpson et.al)
- complete Segal spaces (Rezk)
- categories with weak equivalences (Dwyer-Kan, Barwick-Kan)

Simplicial spaces: $X: \Delta^{op} \to sSets$.

Definition: A segal space X is a simplicial space such that the Segal maps

$$X_n \to \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n}$$

are weak equivalences for all $n \geq 2$.

(The maps above are induced by

$$\alpha^i : [1] \to [n] \quad 0 \mapsto i, \quad 1 \mapsto i+1$$

in Δ:).

In this definition, the objects are $X_{0,0}$.

Also have mapping spaces, homotopy equivalences, ...

$\hookrightarrow X_{heq} \subseteq X_1$

Two ways to get $(\infty,1)$-categories:

1) Ask that X_0 discrete $\rightsquigarrow$ Segal category

2) Ask that $X_{heq} \simeq X_0$ $\rightsquigarrow$ complete Segal spaces.

Have model categories for both:

1) due to Pelissier, Bergner

2) due to Rezk.

Theorem: These model categories are Quillen equivalent to the model ~~categories~~ category of simplicial categories.

The complete segal space model structure is cartesian, so we ~~~~ can enrich in complete segal spaces structure $\rightsquigarrow$ model cat. of $(\infty,2)$-structure.

But we cannot enrich again (run into the same problem).

Want: Higher version of complete segal spaces.

Tool: Θ- construction (due to Berger).

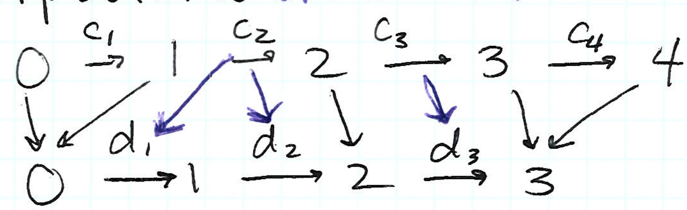
start with a category $\mathcal{C}$.

Define $\Theta\mathcal{C}$ to have objects.

$$\underset{\text{object of } \Delta}{[m]}(\underbrace{c_1, \dots, c_m}_{\text{objects of } \mathcal{C}}).$$

A visualization: $0 \xrightarrow{c_1} 1 \xrightarrow{c_2} \dots \to m-1 \xrightarrow{c_m} m.$

and morphisms in blue:



no morphisms out of c_1, c_4.

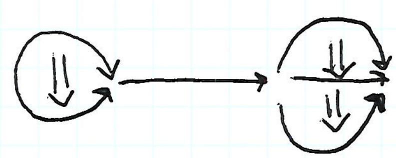
Define $\Theta_0 = *$, $\Theta_n = \Theta\Theta_{n-1}$.

Note: $\Theta_1 = \Delta$

what about Θ_2? $\Theta_2 = \Theta\Delta$

Example: $[3]([1],[0],[2])$

$$0 \xrightarrow{[1]} 1 \xrightarrow{[0]} 2 \xrightarrow{[2]} 3$$



This looks like a model of a 2-category.

Definition: A Θ_n-space is a functor $X : \Theta_n^{op} \to \mathrm{sSets}$ satisfying segal and completeness conditions at all levels.

Theorem (Rezk): There is a cartesian model structure for Θ_n-spaces.

An example of a Segal condition:

$$X\left(\bullet \overset{\Downarrow}{\rightrightarrows} \bullet \longrightarrow \bullet \overset{\Downarrow}{\underset{\Downarrow}{\Rrightarrow}} \bullet\right) \simeq X\left(\bullet \overset{\Downarrow}{\rightrightarrows} \bullet\right) \times_{X(\cdot)} X(\to) \times_{X(\cdot)} \underbrace{X\left(\bullet \overset{\Downarrow}{\underset{\Downarrow}{\Rrightarrow}} \bullet\right)}_{\cong\; X\left(\bullet \overset{\Downarrow}{\rightrightarrows} \bullet\right) \times_{X(\to)} X\left(\bullet \overset{\Downarrow}{\rightrightarrows} \bullet\right)}$$

Conjecture: (Θ_{n-1}-spaces)-cat is Quillen equivalent to Θ_n-Sp.

$\underline{n=1}$: $(\mathrm{sSet})\text{-cat} \rightleftarrows \mathrm{sSets}^{\Delta^{op}}_{se,\,disc} \rightleftarrows \mathrm{sSets}^{\Delta^{op}}_{se,\,cpt}$

General: $\underbrace{(\Theta_{n-1}\text{-Sp})\text{-Cat} \rightleftarrows (\Theta_{n-1}\mathrm{Sp})^{\Delta^{op}}_{se,disc}}_{\text{Thm of Bergner, Rezk}} \underbrace{\rightleftarrows (\Theta_{n-1}\mathrm{Sp})^{\Delta^{op}}_{se,cpt} \rightleftarrows \Theta_n\mathrm{Sp}}_{\text{work in progress.}}$

Have also: $(\Theta_{n-1}\mathrm{Sp})^{\Delta^{op}}_{se,cpt} \underset{\leftarrow}{\overset{?}{\rightarrow}} \Theta_n\mathrm{Sp}$

$$(\Theta_{n-1}\mathrm{Sp})^{\Delta^{op}}_{se,cpt} = (\mathrm{ssets})^{\Delta^{op} \times \Theta_{n-1}^{op}}_{\sim}, \qquad \Theta_n\mathrm{Sp} = (\mathrm{ssets})^{\Theta_n^{op}}_{\sim}$$

How to think about completeness conditions?

$X_{heq} \simeq X_0$, $\quad X(\overset{\sim}{\to}) \simeq X(\cdot)$, $\quad X(\to) \simeq X\left(\bullet \overset{\Downarrow}{\rightrightarrows} \bullet\right)$.