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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Leanne Merrill **Email/Phone:** leannem@uoregon.edu / 518 461 7614

Speaker's Name: Mark Behrens

Talk Title: Computations in the stable homotopy groups of spheres

Date: 1 / 29 / 14 **Time:** 9 : 00 [✓ am] / pm (circle one)

List 6-12 key words for the talk: Hopf invariant, Mahowald Invariant, Adams Spectral Sequence, Atiyah-Hirzebruch spectral Sequence, Adams-Nobikov Spectral sequence, Kervaire invariant.

Please summarize the lecture in 5 or fewer sentances: This talk surveyed various methods for computing homotopy groups of spheres. It used spectral sequence techniques and invariants to show classical results at the prime 2. It ended by stating a conjecture about periodic elements in the chromatic spectral sequence.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
- **Overhead**: Obtain a copy or use the originals and scan them
- **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
- **Handouts**: Obtain copies of and scan all handouts

☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

☑ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Mark Behrens   <u>Computations in the Homotopy Groups of Spheres</u>

1/29/14

This is a Powerpoint talk, and these notes are intended to supplement the Powerpoint slides.

Where do infinite groups appear?

$$\begin{tikzcd}
\pi_{4n-1}(S^{2n}) \arrow[r, "\text{Hopf invariant, HI}"'] & \mathbb{Z} \\
\alpha \arrow[u, phantom, "\rotatebox{90}{$\in$}"] &
\end{tikzcd}$$

$H^* C\alpha$ $\quad \mathbb{Z}[y] \quad 4n$

$S^{4n-1} \to S^{2n} \to C\alpha$ $\quad \mathbb{Z}[x] \quad 2n$

$$x^2 = \pm HI(\alpha)\, y.$$

$HI(\alpha) = 1$?

$(\mathbb{R})$ $\quad S^1 \xrightarrow{2} S^1$

$(\mathbb{C})$
$$\begin{tikzcd}
S^3 \arrow[r, "\eta"] \arrow[d, phantom, "\cong" rotate=90] & S^2 \arrow[d, phantom, "\cong" rotate=90] \\
\mathbb{C}^2 \setminus \{0\} \arrow[r] & \mathbb{C}P^1
\end{tikzcd}$$

$(\mathbb{H})$ $\quad S^7 \xrightarrow[\nu]{} S^4$

$(\mathbb{O})$ $\quad S^{15} \xrightarrow[\sigma]{} S^8$

<u>Thm</u> (Adams '60)

These are the only dimensions with $HI = 1$.

Stable homotopy Groups of spheres at the prime 2.

A pattern shows up (highlighted).

Why?

$$SO(n) \to \Omega^n S^n$$

$$A : \mathbb{R}^n \to \mathbb{R}^n \longmapsto A^+$$

$$\begin{tikzcd} & \pi_*^s \arrow[d, phantom, "\shortparallel"] & \\ \pi_* SO \arrow[r, "J"'] & \pi_* QS^0 \arrow[r, phantom, "="] & \Omega^\infty \Sigma^\infty S^0 \end{tikzcd}$$

"the J homomorphism"

Show up with the pattern.

$$\mathbb{Z}_2 \; 0 \; \mathbb{Z} \; 0 \; 0 \; 0 \; \mathbb{Z} \; \mathbb{Z}/2 \; \mathbb{Z}/2 \; 0 \; \mathbb{Z} \; 0 \; 0 \; 0 \; \mathbb{Z} \; \cdots$$

Theorem (Adams '66) $\mathbb{Z}/2 \longmapsto$ nontrivial

$$\# \left| \operatorname{Im} J_{4k-1} \right| = \operatorname{denom}\left(\frac{B_{2k}}{4k} \right)$$

where B_j denotes a Bernoulli number.

EHP Sequence:

$$\Omega^{n+1} S^{2n+1} \xrightarrow[P]{} \Omega^{n-1} S^{n-1} \xrightarrow[E]{} \Omega^n S^n \xrightarrow[H]{} \Omega^n S^{2n-1}$$

related to Whitehead product

(suspension)

(Hopf invariant)

$$\pi_{2n-1} S^n \xrightarrow[HI]{} \pi_{2n-1} S^{2n-1} = \mathbb{Z}.$$

forms a fiber sequence.

Splice together to get EHP Spectral Sequence.

$$E_1 = \pi_{n+k} S^{2n-1} \underset{\text{converges to}}{\Longrightarrow} \pi_k^s.$$

Method for computing differentials in EHP S.S.:

$$\begin{tikzcd}
\Omega^n S^n \arrow[r, "E"'] \arrow[d] & \Omega^{n+1} S^{n+1} \arrow[r, "H"'] \arrow[d] & \Omega^{n+1} S^{2n+1} \arrow[d] \\
Q\,\mathbb{R}P^{n-1}_+ \arrow[r] & Q\,\mathbb{R}P^n_+ \arrow[r] & Q\,S^n
\end{tikzcd}$$

gives a map of spectral sequences, from EHP spectral sequence to Atiyah-Hirzebruch s.s.

$$\begin{tikzcd}
\bigoplus_{n\geq 0} \pi_*(S^{2n+1}) \arrow[r, Rightarrow, "EHPSS"] \arrow[d] & \pi_*^s \arrow[d] \\
\bigoplus_{n\geq 0} \pi_*^s(S^n) \arrow[r, Rightarrow, "AHSS"'] & \pi_*^s(\mathbb{R}P^\infty_+)
\end{tikzcd}$$

In
$$\begin{tikzcd}
E_1 = \pi_{n+k} S^{2n-1} \arrow[r, Rightarrow] & \pi_k^s \\
HI(\alpha) \arrow[u, phantom, sloped, "\in"] \arrow[r, Rightarrow] & \alpha \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$
a generalized Hopf invariant.

Another invariant: the Mahowald invariant.

Can extend the periodic differentials to get a map

$$\begin{tikzcd}
\bigoplus_{n\in\mathbb{Z}} \pi_*^s(S^n) \arrow[r, Rightarrow] & \pi_*^s \mathbb{R}^\infty_{-\infty} \arrow[d, phantom, sloped, "\cong"] \\
MI(\alpha) \arrow[u, phantom, sloped, "\in"] \arrow[rd, Rightarrow] & \pi_*^s(S^{-1}) \\
& \alpha \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

} due to Lin '80.

the Mahowald invariant.

Have $\deg(HI(\alpha)) < \deg(\alpha)$

$\deg(MI(\alpha)) > \deg(\alpha)$.

We know:

$MI(2) = \eta$

$MI(\eta) = \nu$

$MI(\nu) = \sigma$.

$MI(\sigma) = \sigma^2$.

Best way to compute stable homotopy groups of spheres is the Adams spectral sequence.

Let E = "nice" ring spectrum so that (E_*, E_*E) = Hopf algebroid.

This means that there is a comultiplication

$$E_*E \longrightarrow E_*E \otimes_{E_*} E_*E \quad \text{(co algebra)}$$

$$E_*X \longrightarrow E_*E \otimes_{E_*} E_*X \quad \text{(co module).}$$

The E-based Adams spectral sequence:

$$E_2 = \mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*X) \Rightarrow \pi_{t-s} X^\wedge_E$$

The case where $E = H\mathbb{F}_p$, $E_*E = A_*$ is just called the Adams Spectral Sequence. it was the first historical case.

In general, $HI(x) = 1 \iff x$ detected by h_j

$d_2(h_j) = h_{j-1}^2 h_0 \quad j \geq 4$ (this is the main key to the Hopf invariant 1 theorem).

Kervaire invariants are detected on the 2-line.

What is the Kervaire invariant?

$\pi_*^s \cong \Omega_*^{fr}$ $\qquad$ Θ_n = group of exotic n-spheres.

$$\Theta_n \longrightarrow \pi_n^s / \mathrm{Im}\, J$$

Kervaire-Milnor show that if $n \not\equiv 2 \bmod 4$, the above map is surjective.

But if $n = 2 \bmod 4$, there is an obstruction, which is the Kervaire invariant.

$$\Theta_n \to \pi_n^s / \mathrm{Im}\, J \xrightarrow[K.I.]{} \mathbb{Z}/2 .$$

<u>Thm</u> (Browder '69).

$$KI(x) = 1 \iff x \text{ detected by } h_j^2 \text{ in ASS.}$$

But h_j^2 is a permanenct cycle for $j \leq 5$.

The case $j = 5$ was computed by Barrell-Jones-Mahowald in '80.

But, in fact,

Hill-Hopkins-Ravenel '09:

h_j^2 not a permanent cycle $j \geq 7$.

Question: h_6^2 a p.c.?

Question: $d_r(h_j^2) = ?$

Theorem (Mahowald '77)

$h_1 h_j$ are all permanent cycles for $j \geq 2$.
$$\begin{matrix} h_1 h_j \\ \shortparallel \\ \eta_j \end{matrix} .$$

Mahowald - Ravenel: $HI(h_j x) = MI(x)$.

$$\text{So:} \quad HI(\eta_j) = HI(h_1 h_j) = MI(\eta) = \nu$$

This is an example of infinitely many elements with the same Hopf invariant.

Question: Is $h_0 h_j$ a permanent cycle for $p \geq 2$?

Adams - Nobiko spectral sequence:

$E = BP$

$\pi_* BP = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$

$|v_i| = 2(p^i - 1)$.

In ANSS, Kervaire invariants are detected by $\beta_{2^i/2^i}$.

ANSS has far fewer differentials at $p = 3$.

Thm (Miller '81):

$$\begin{tikzcd} E_2^{ASS} \arrow[r] \arrow[rr, bend right, "ASS"'] & \underset{0}{E_*^{ANSS}} \arrow[r, Rightarrow, "ANSS"'] & \pi_*^S \end{tikzcd} .$$

Thm (Toda '68):

$$d_r(\beta_{p/p}) = \beta_1^p \alpha_1 \quad \text{for } p \text{ odd.}$$

in ANSS.

we call β_{p^i/p^i} odd primary K.I. elements.

Thm (Ravenel '78) For $p \geq 5$.

β_{p^i/p^i} is not a permanent cycle for $i \geq 1$.

Question is $\beta_{3^i/3^i}$ a P.C.?

Chromatic spectral sequence (Miller-Ravenel-Wilson '77)

"v_n-periodic"

$$\bigoplus_{n \geq 0} \mathrm{Ext}_{BP_*BP}\left(BP_*, \frac{BP_*}{(p^\infty, v_1^\infty, \dots, v_{n-1}^\infty)}[v_n^{-1}]\right)$$

$$\Longrightarrow \mathrm{Ext}_{BP_*BP}(BP_*, BP_*) = E_2^{ANSS}$$

The advantage is that the top groups are, in principle, computable.

In $(\pi_* S)_{(p)}$, $p=5$:

: $\quad HI(\alpha_{i/j}) = \alpha_{i-j} \qquad MI(p^j) = \alpha_j$

$HI(\beta_{i/j,k}) = \beta_{i-j/k}$ (in progress)

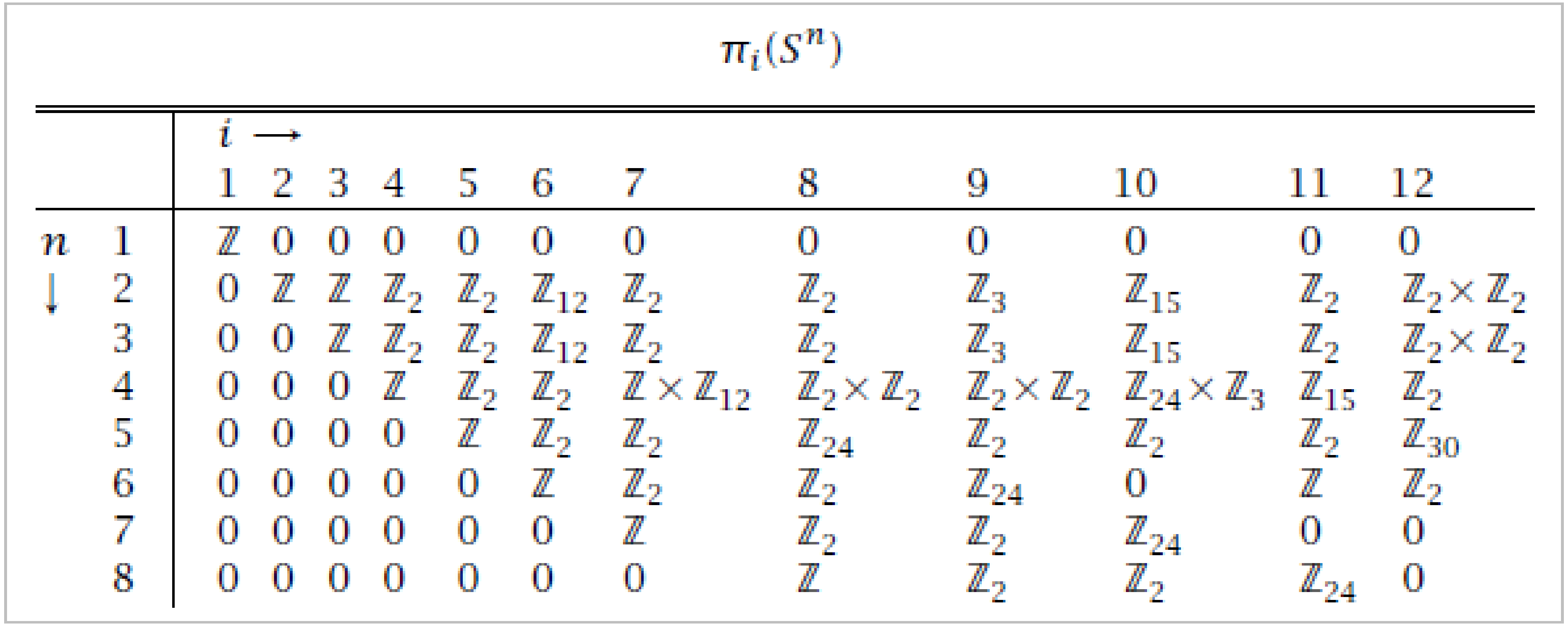
"$MI(\alpha_{i/j}) = \beta_{i/j}$"

Conjecture: MI elements go from one periodic layer to the next.

Computations in the homotopy groups of spheres

Mark Behrens

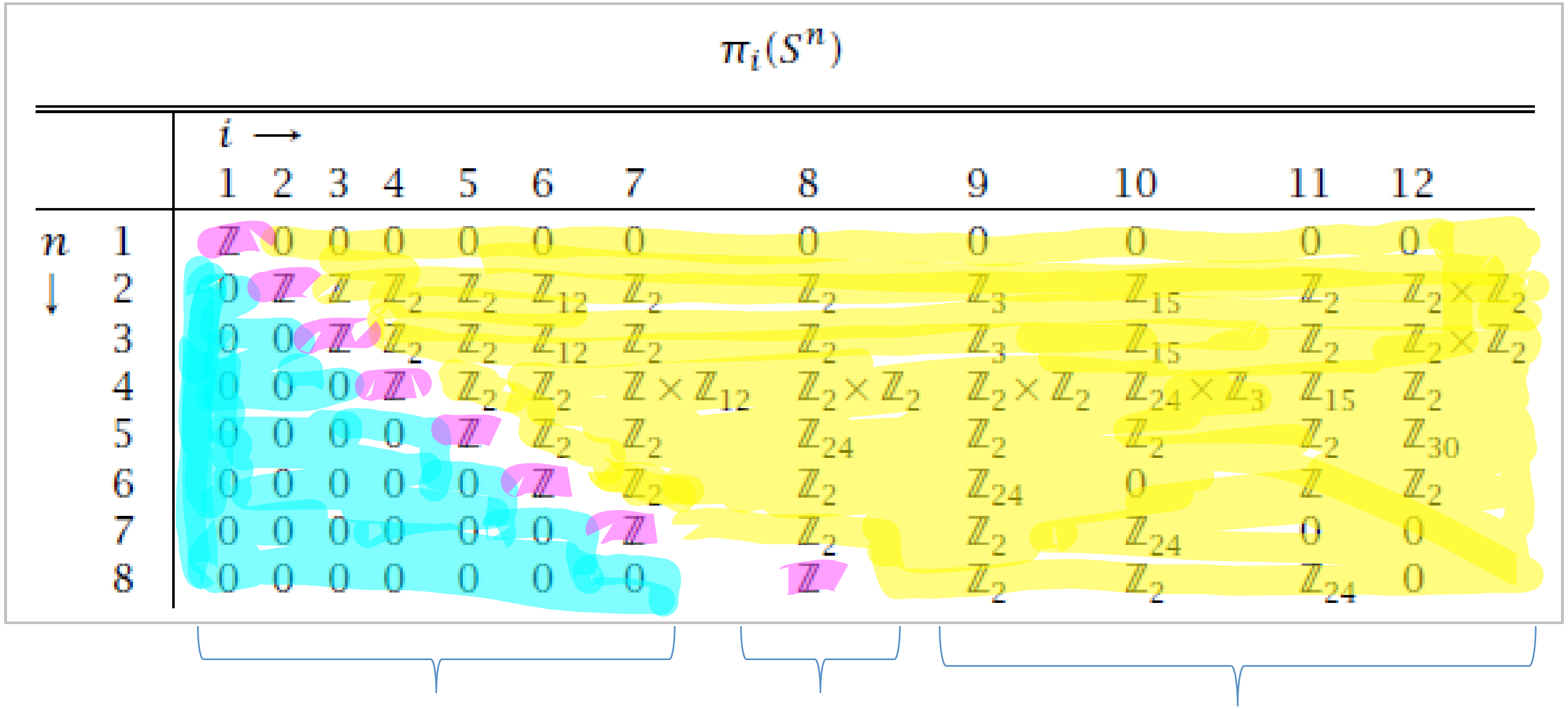
(MIT)



$\pi_i(S^n)$

		$i \longrightarrow$											
		1	2	3	4	5	6	7	8	9	10	11	12
n	1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
↓	2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
	3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
	4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$
	5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
	6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
	7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0
	8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

Homotopy groups of spheres in low dimensions



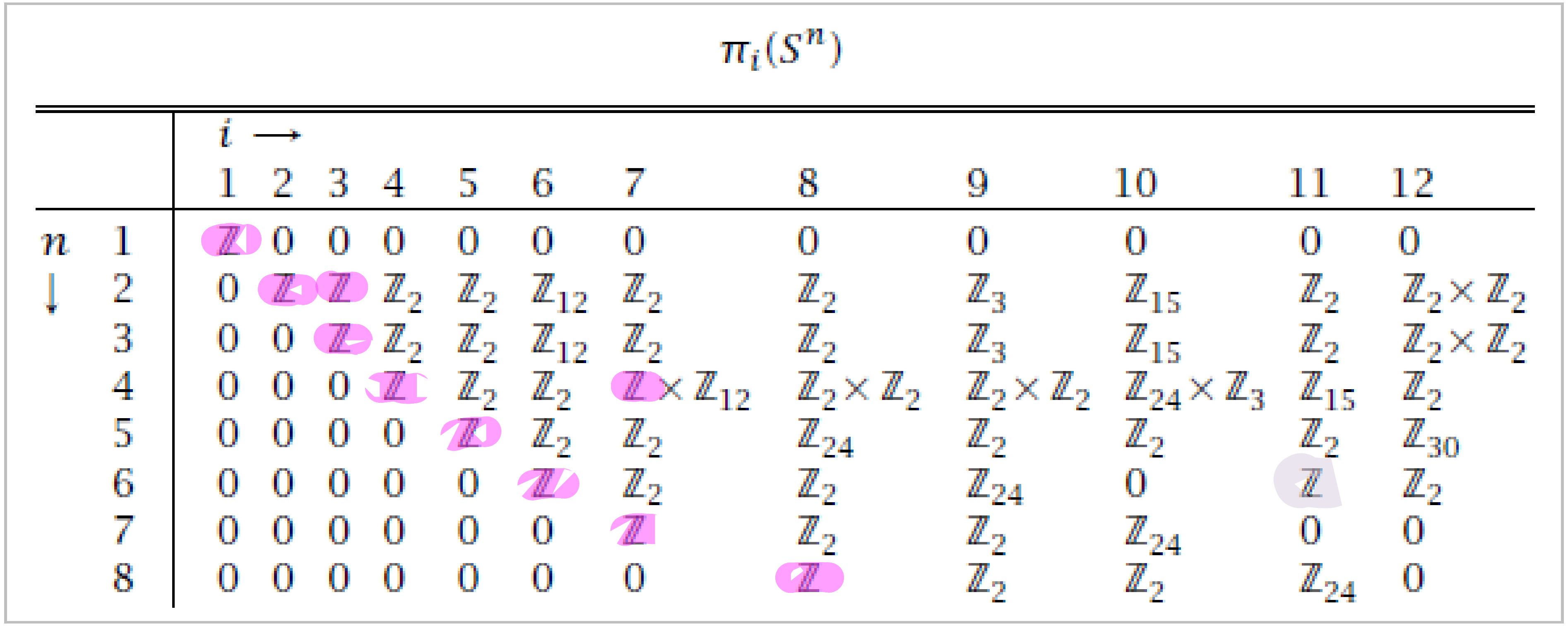
$\pi_i(S^n)$

		$i \longrightarrow$											
		1	2	3	4	5	6	7	8	9	10	11	12
n	1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
↓	2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
	3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
	4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$
	5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
	6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
	7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0
	8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

$\pi_{<k}(S^k)$

$\pi_k(S^k)$

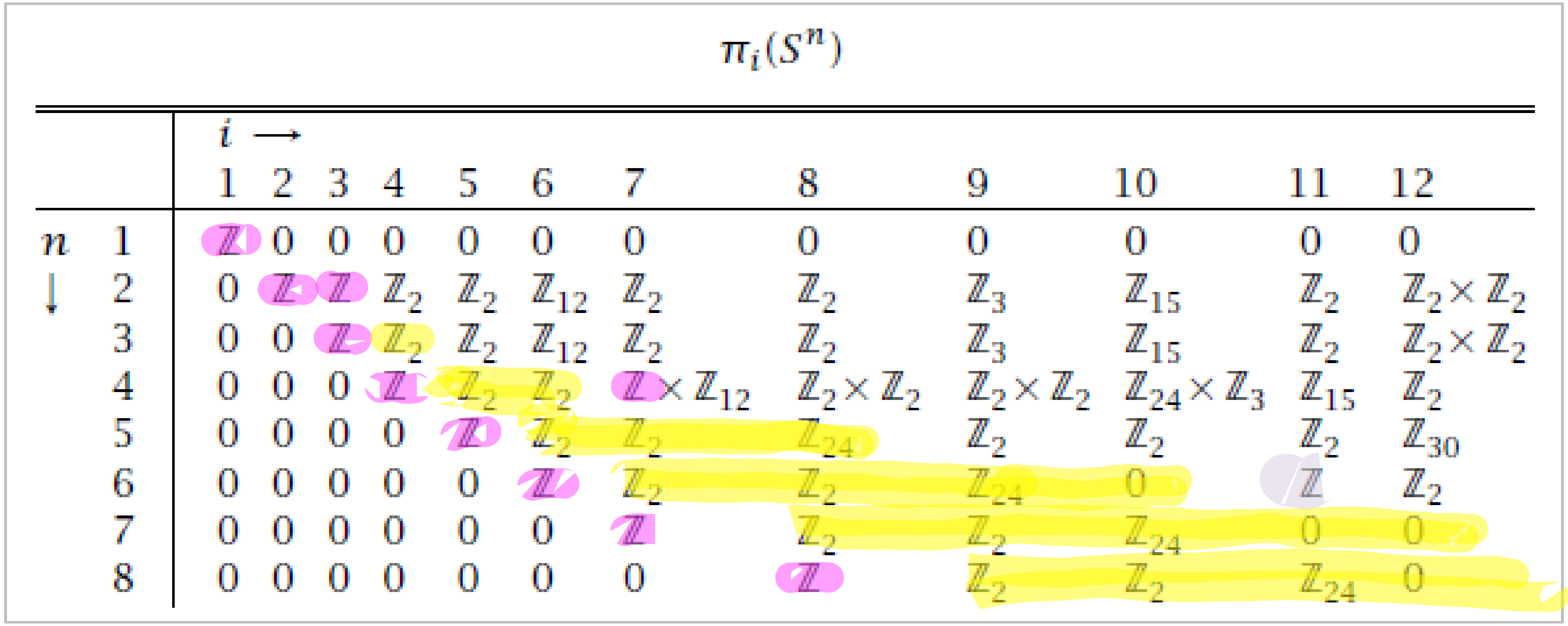
$\pi_{>k}(S^k)$



$\pi_i(S^n)$

		$i \longrightarrow$											
		1	2	3	4	5	6	7	8	9	10	11	12
n	1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
$\downarrow$	2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
	3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
	4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$
	5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
	6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
	7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0
	8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

Infinite subgroups completely understood

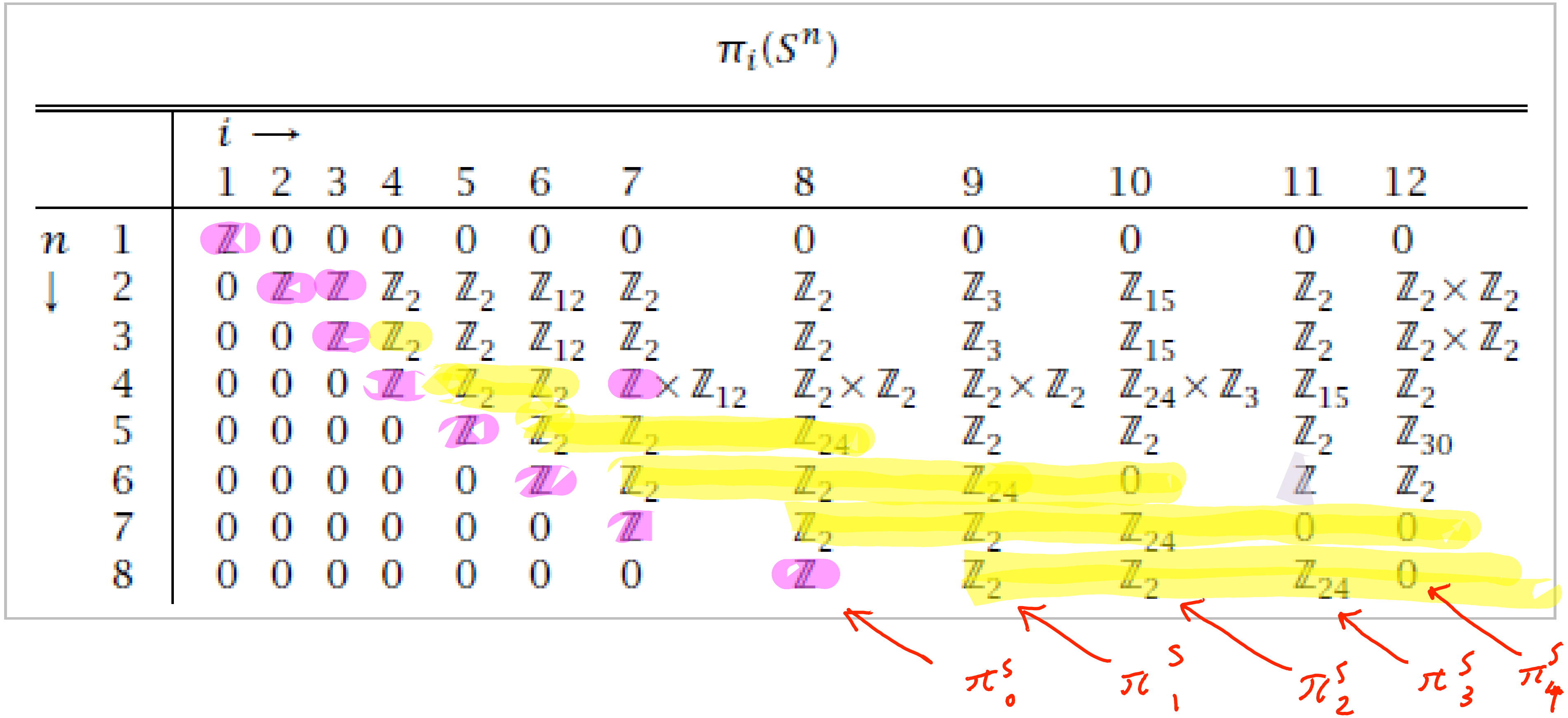


$\pi_i(S^n)$

		$i \longrightarrow$											
		1	2	3	4	5	6	7	8	9	10	11	12
n	1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
↓	2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
	3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
	4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$
	5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
	6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
	7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0
	8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

Values stabilize along diagonals:

$$\pi_{n+k}(S^k) = \pi_{n+k+1}(S^{k+1}) \quad \text{for } k >> 0$$



$\pi_i(S^n)$

i →

n ↓	1	2	3	4	5	6	7	8	9	10	11	12
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$
5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0
8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

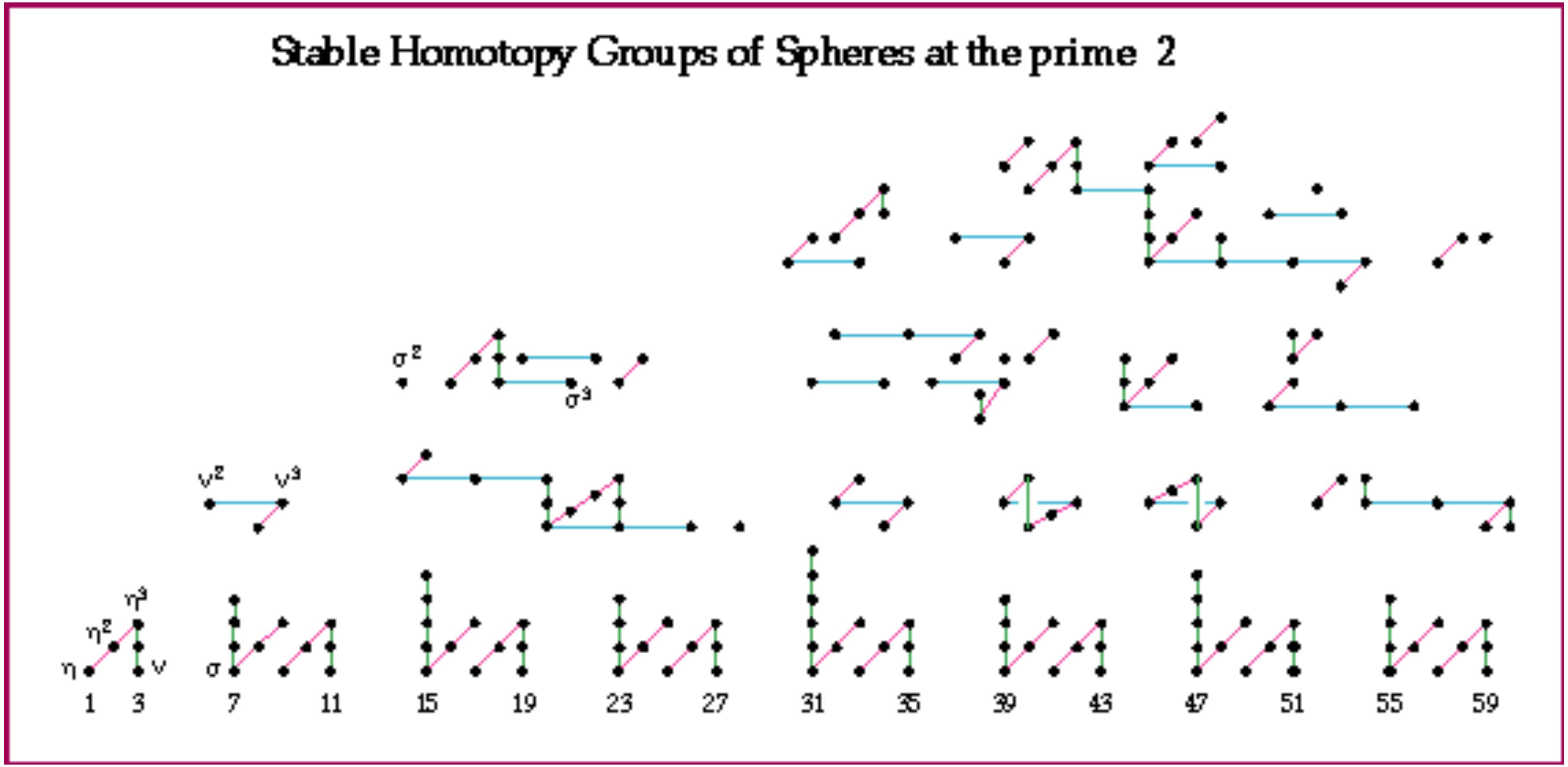
π^s_0 π^s_1 π^s_2 π^s_3 π^s_4

Stable homotopy groups:

$$\pi_n^s := \lim \pi_{n+k}(S^k) \quad k \to \infty$$

Primary decomposition:

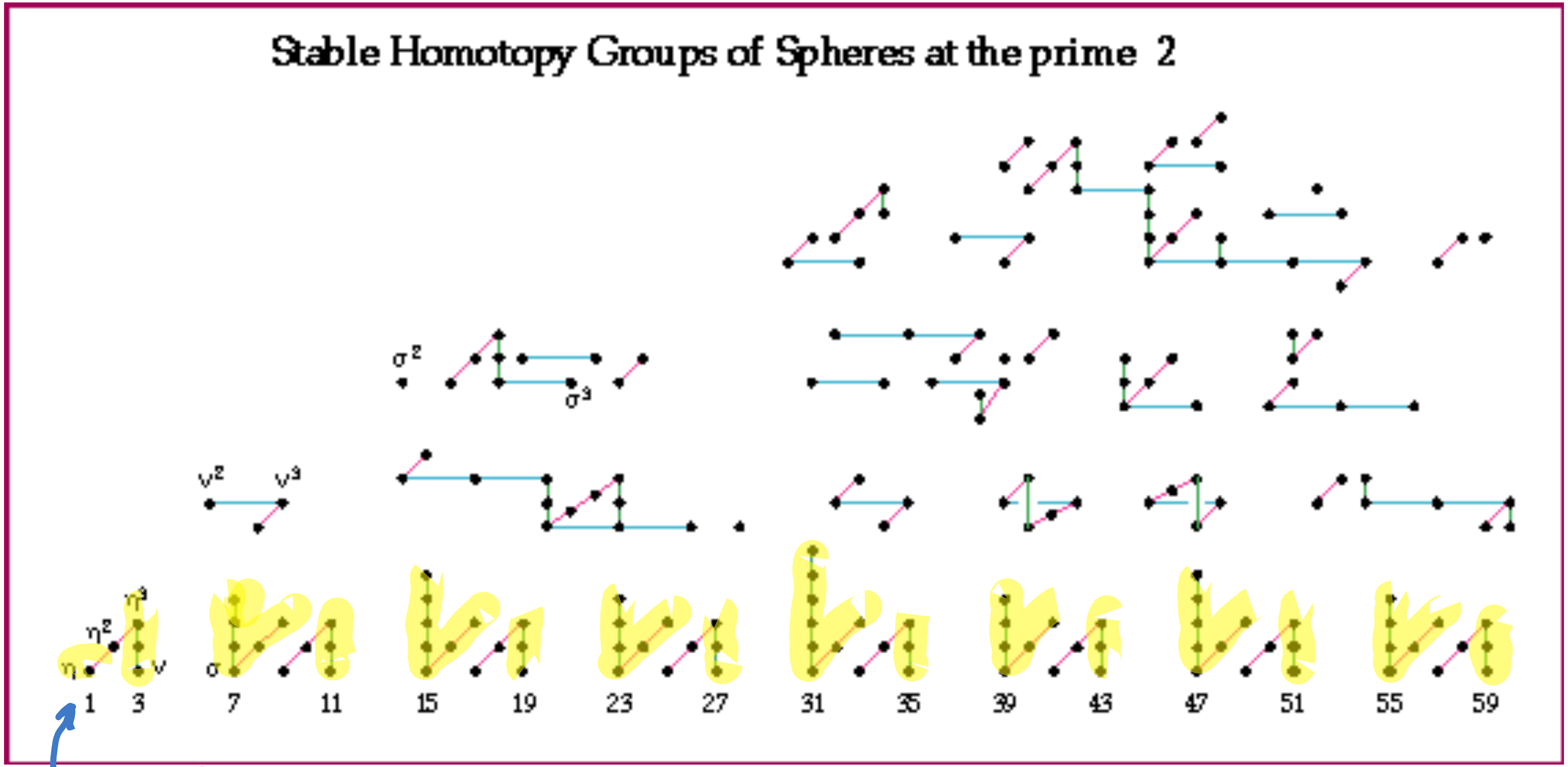
$$\pi_n^s = \bigoplus_{p \text{ prime}} (\pi_n^s)_{(p)} \qquad \text{e.g.:} \qquad \pi_3^s = Z_{24} = z_8 + Z_3$$



Computation: Mahowald-Tangora-Kochman
Picture: A. Hatcher

- Each dot represents a factor of 2, vertical lines indicate additive extensions

  e.g.: $(\pi\downarrow 3\uparrow s)\downarrow(2) = \mathbb{Z}\downarrow 8$, $(\pi\downarrow 8\uparrow s)\downarrow(2) = \mathbb{Z}\downarrow 2 \oplus \mathbb{Z}\downarrow 2$
- Vertical arrangement of dots is arbitrary, but meant to suggest patterns

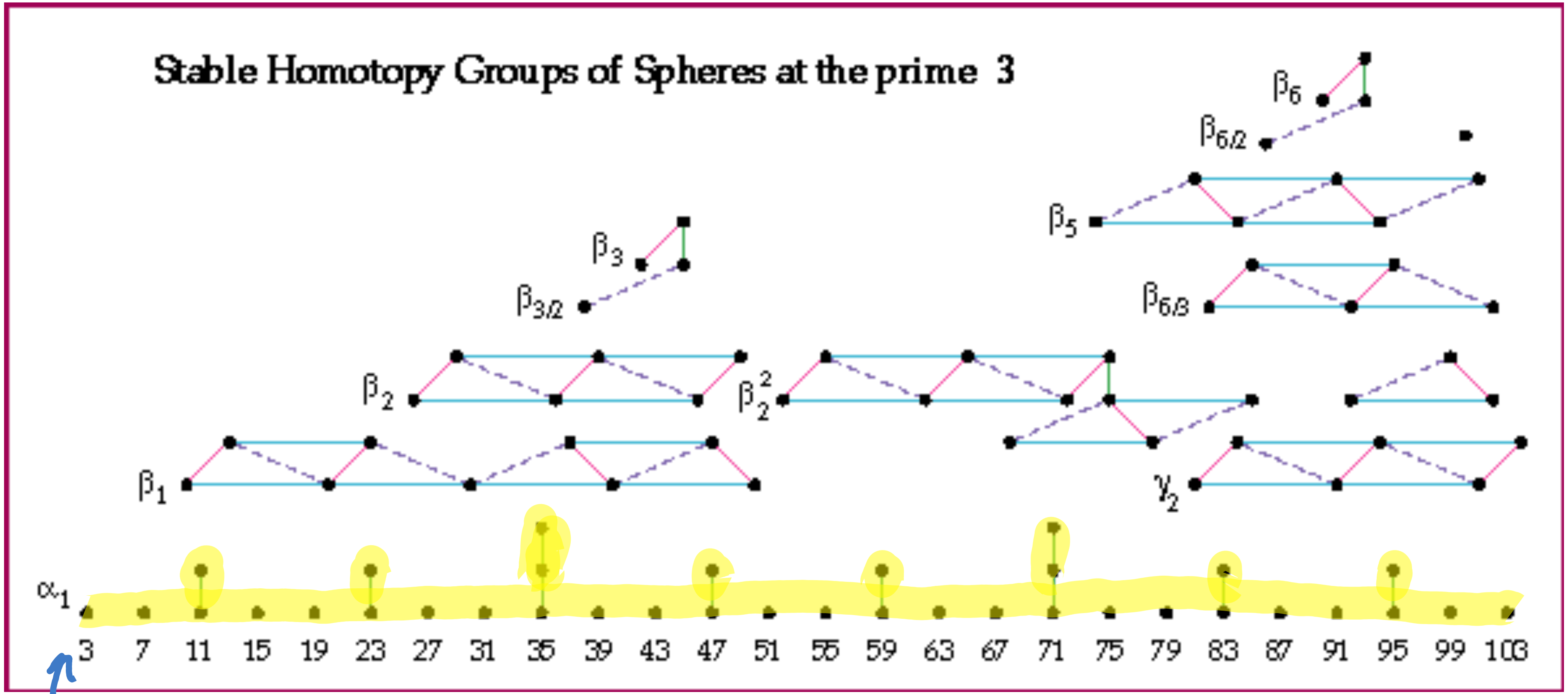


Im J

- Each dot represents a factor of 2, vertical lines indicate additive extensions

  e.g.: $(\pi_3^s)_{(2)} = \mathbb{Z}_8$, $(\pi_8^s)_{(2)} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$
- Vertical arrangement of dots is arbitrary, but meant to suggest patterns

Computation: Nakamura -Tangora
Picture: A. Hatcher



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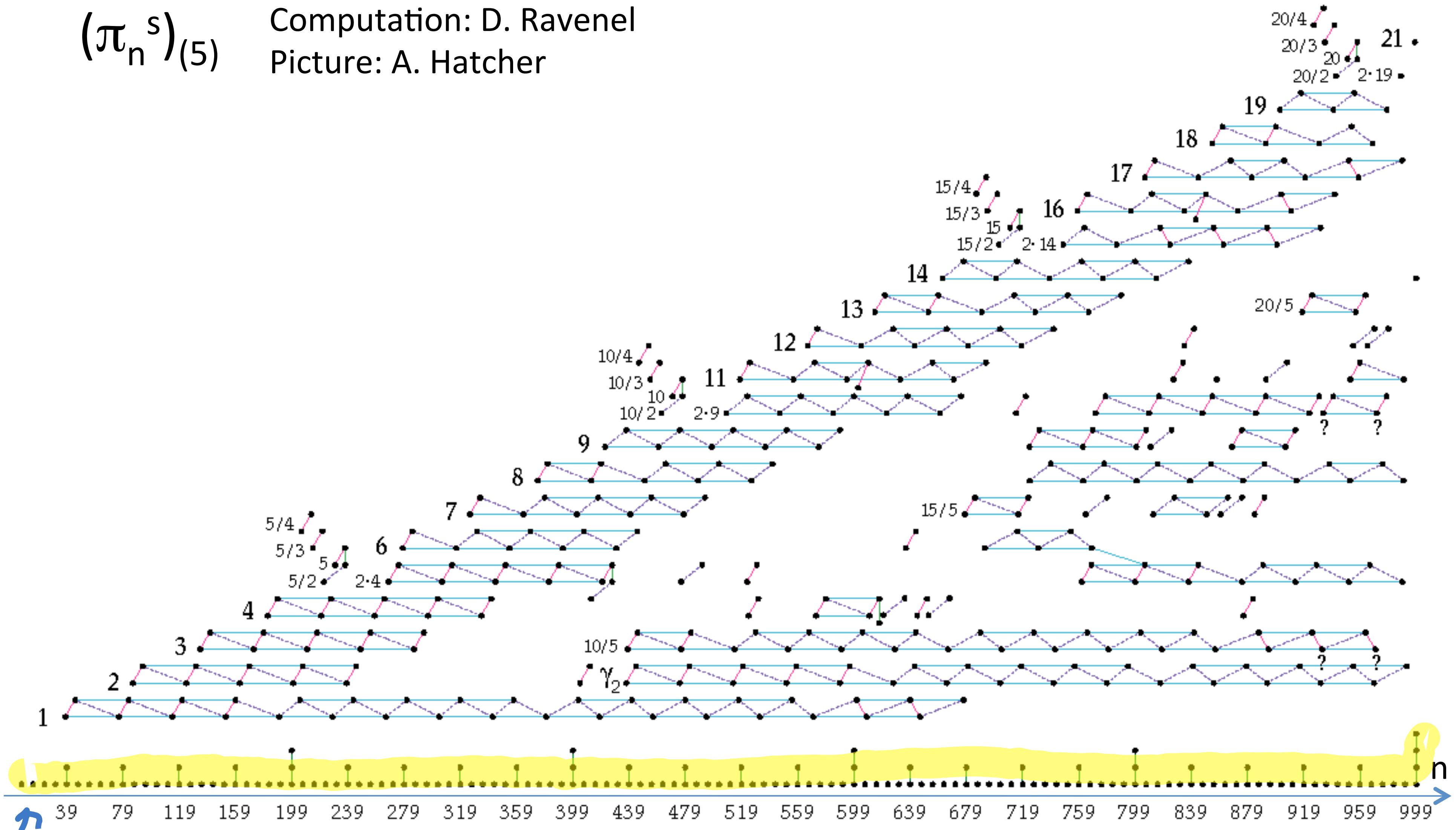
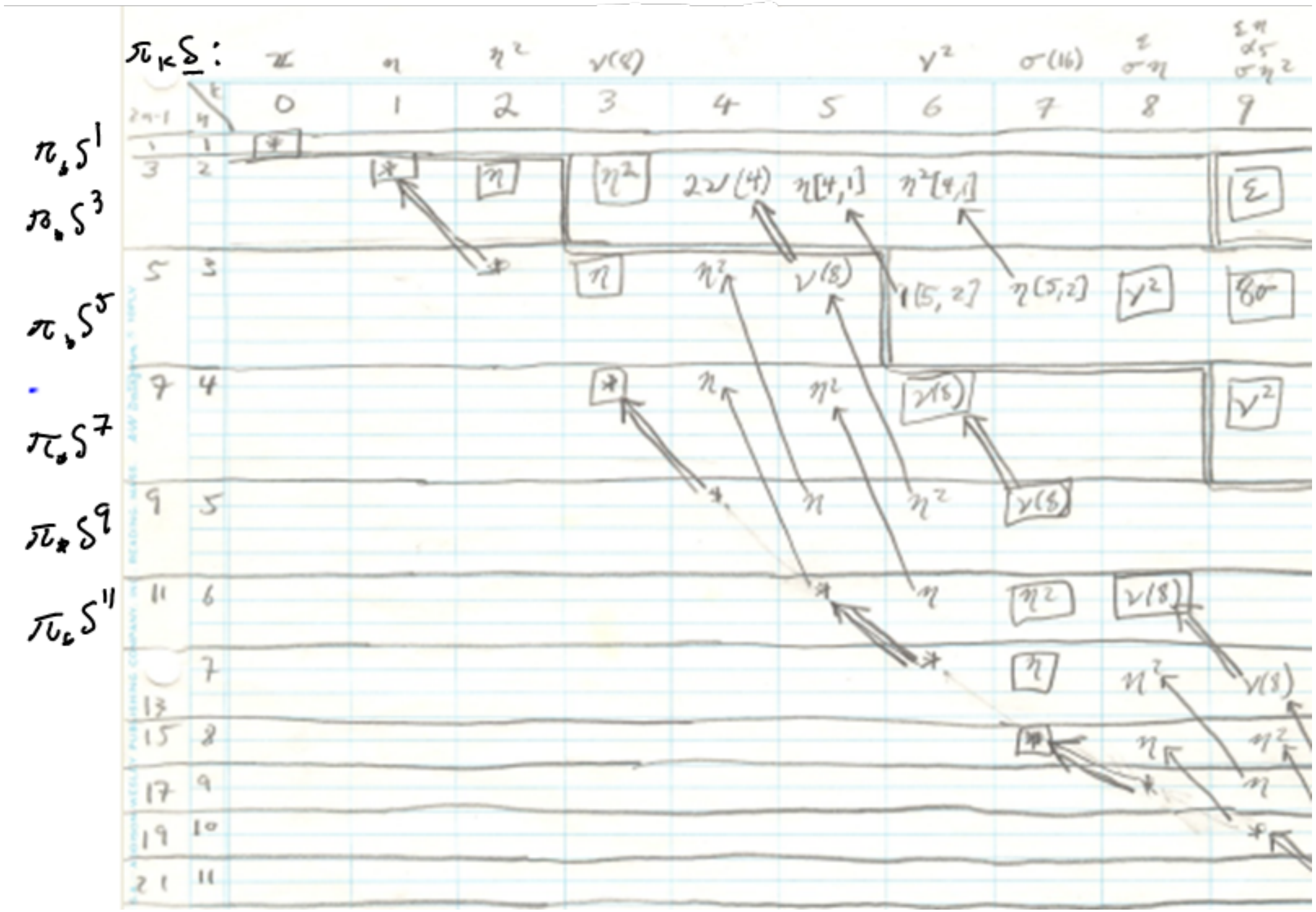
$(\pi_n^s)_{(5)}$ Computation: D. Ravenel
Picture: A. Hatcher



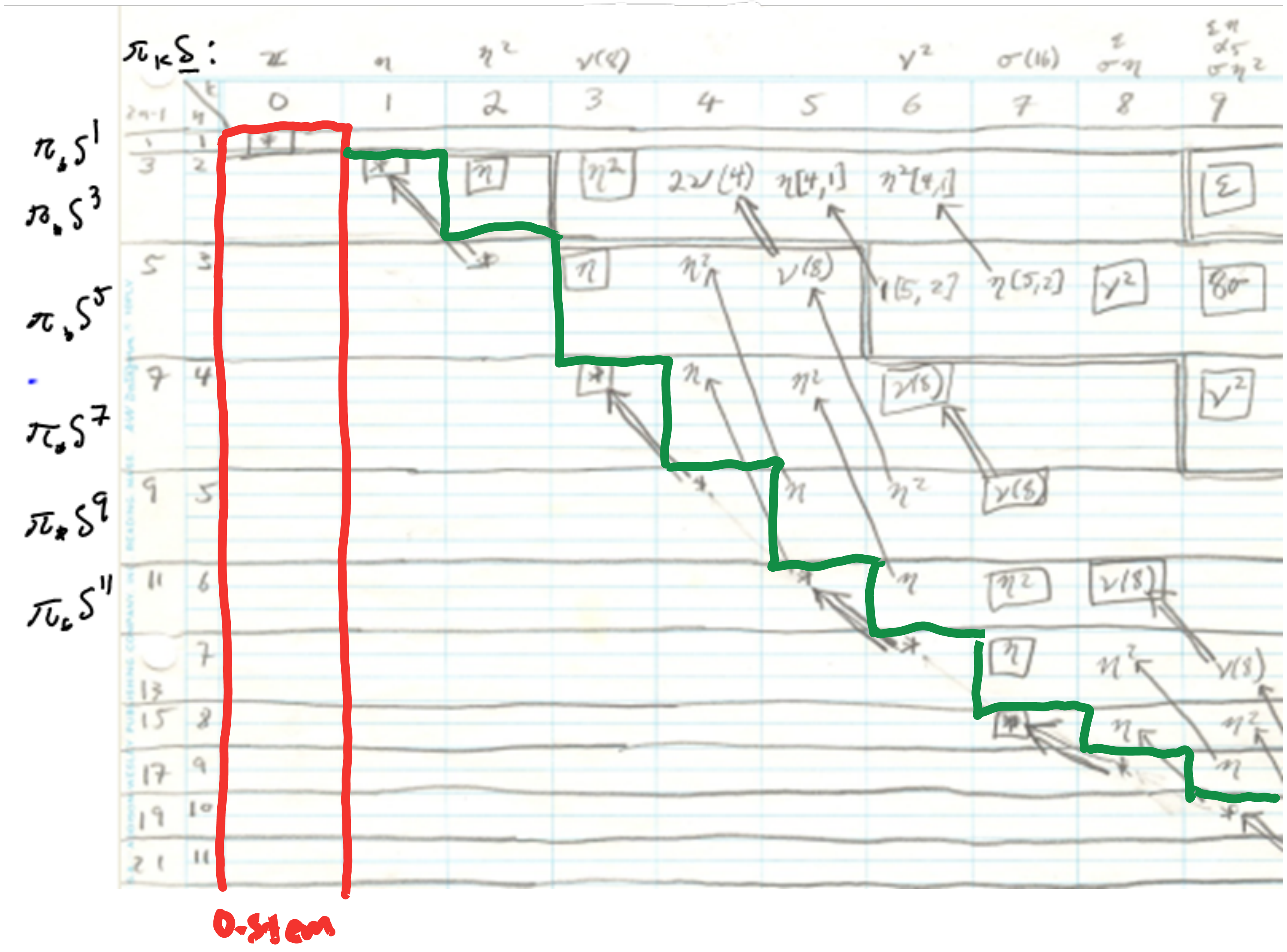
Im J

EHP spectral sequence (p=2)

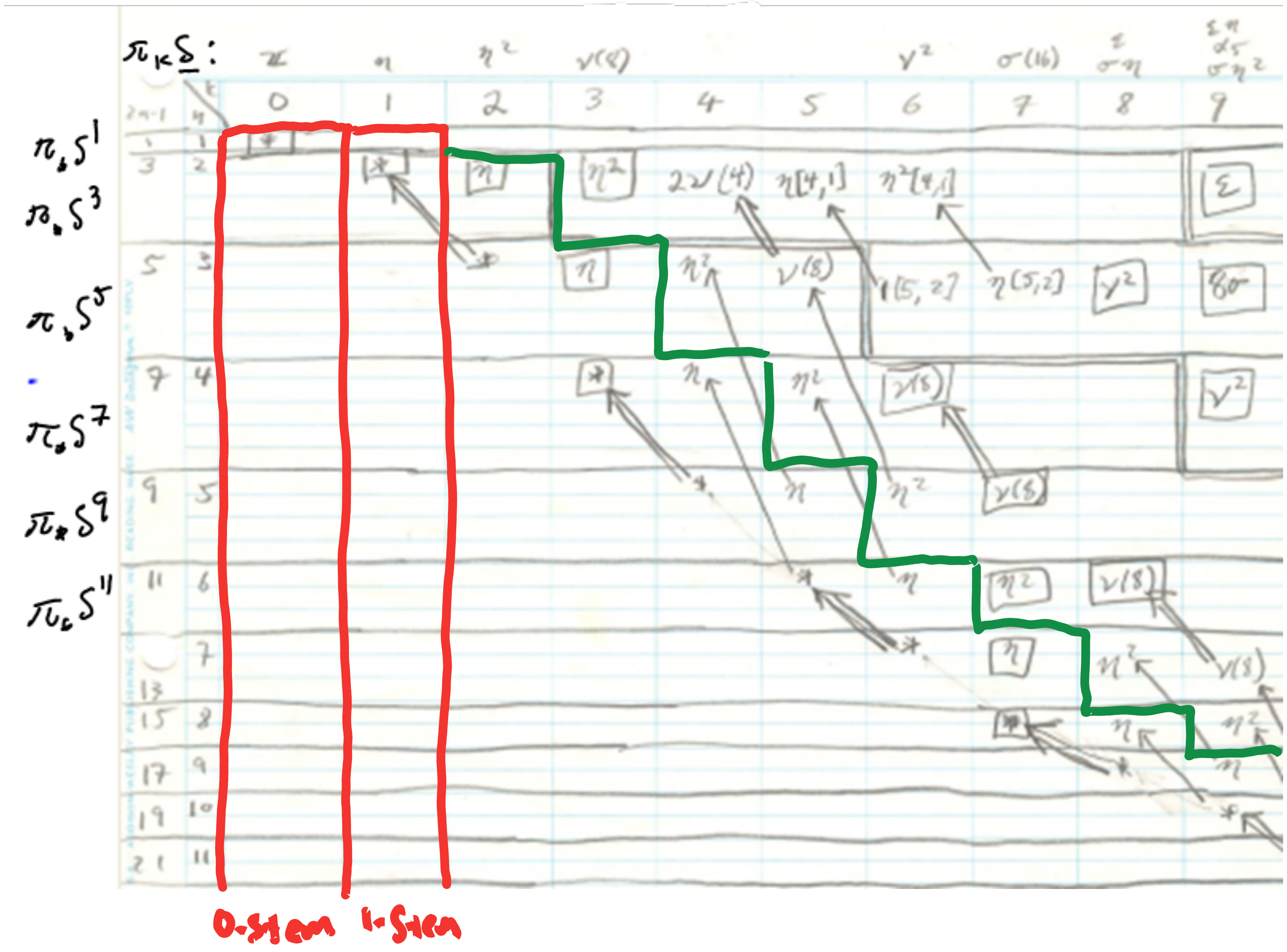


$(* =: \mathbb{Z})$

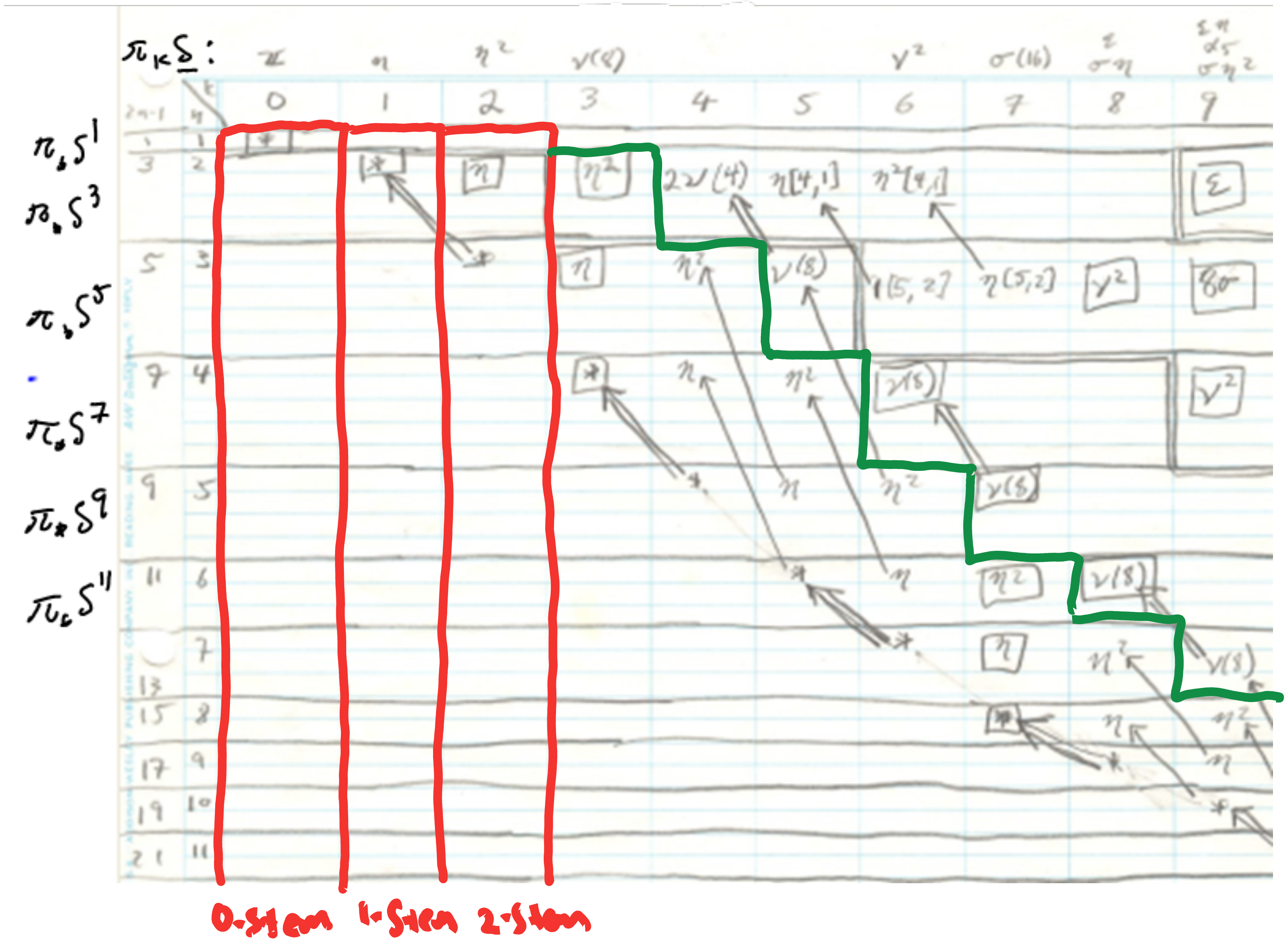
EHP spectral sequence (p=2)

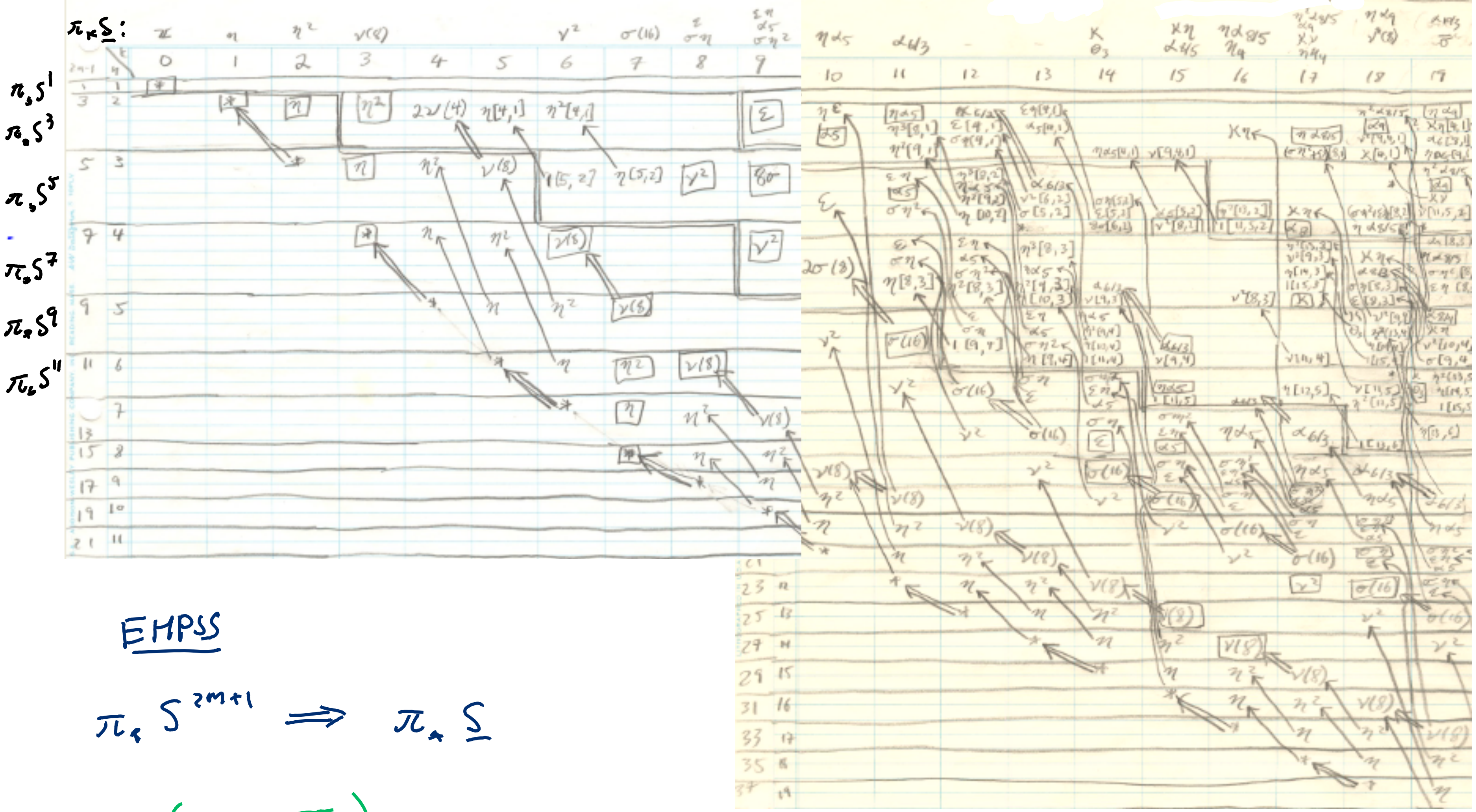


EHP spectral sequence (p=2)



EHP spectral sequence (p=2)

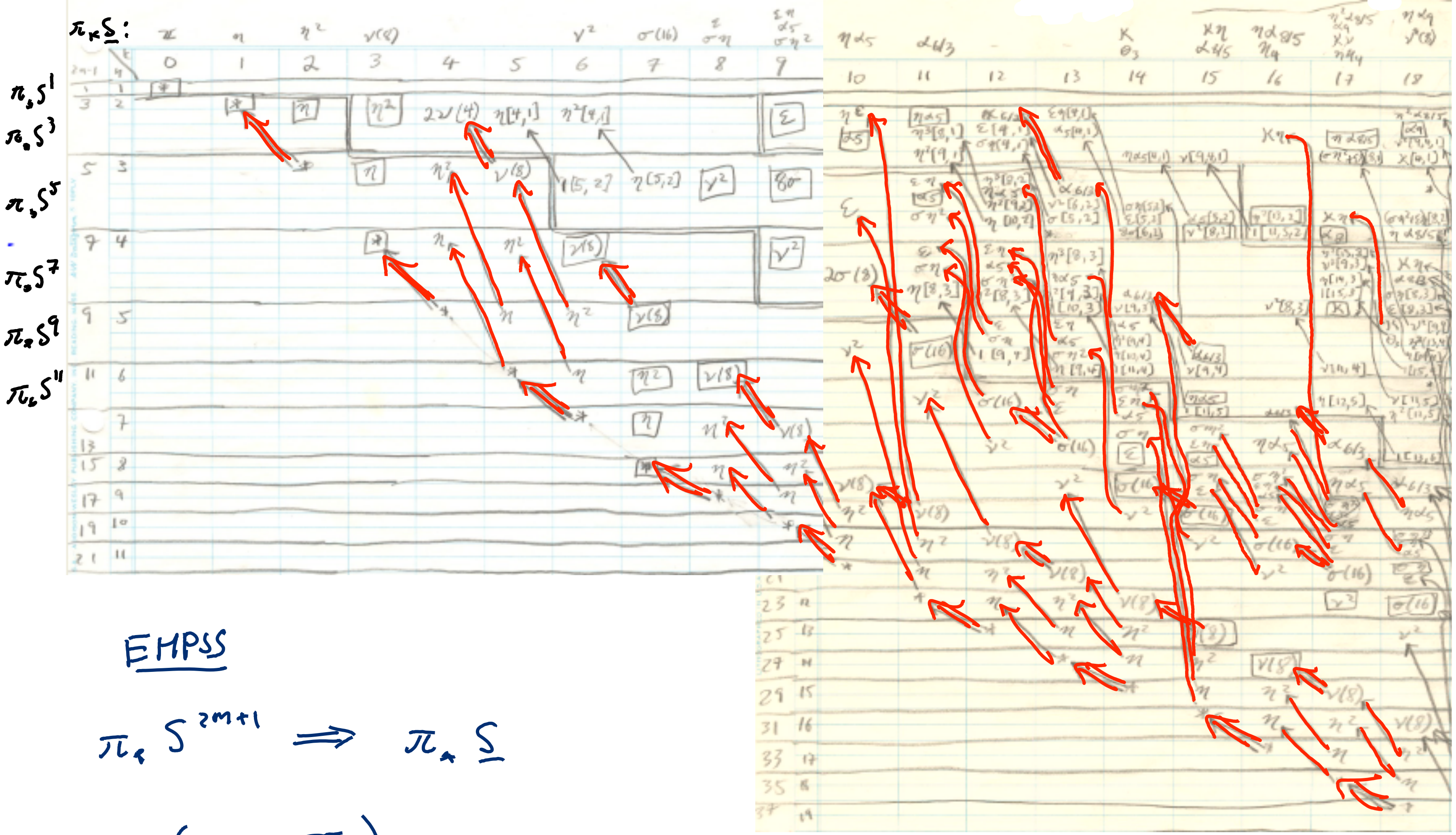




EHPSS

$$\pi_* S^{2m+1} \Longrightarrow \pi_* \underline{S}$$

$$(* =: \mathbb{Z})$$



EHPSS

$$\pi_* S^{2m+1} \Rightarrow \pi_* \underline{S}$$

$(* =: \mathbb{Z})$

diff'ls lifted from AHSS from $\mathbb{R}P^\infty$

Adams spectral sequence

$\mathrm{Ext}_{A}^{s,t}(\mathbb{Z}/p,\mathbb{Z}/p) \Rightarrow (\pi_{t-s}^{s})_{p}$

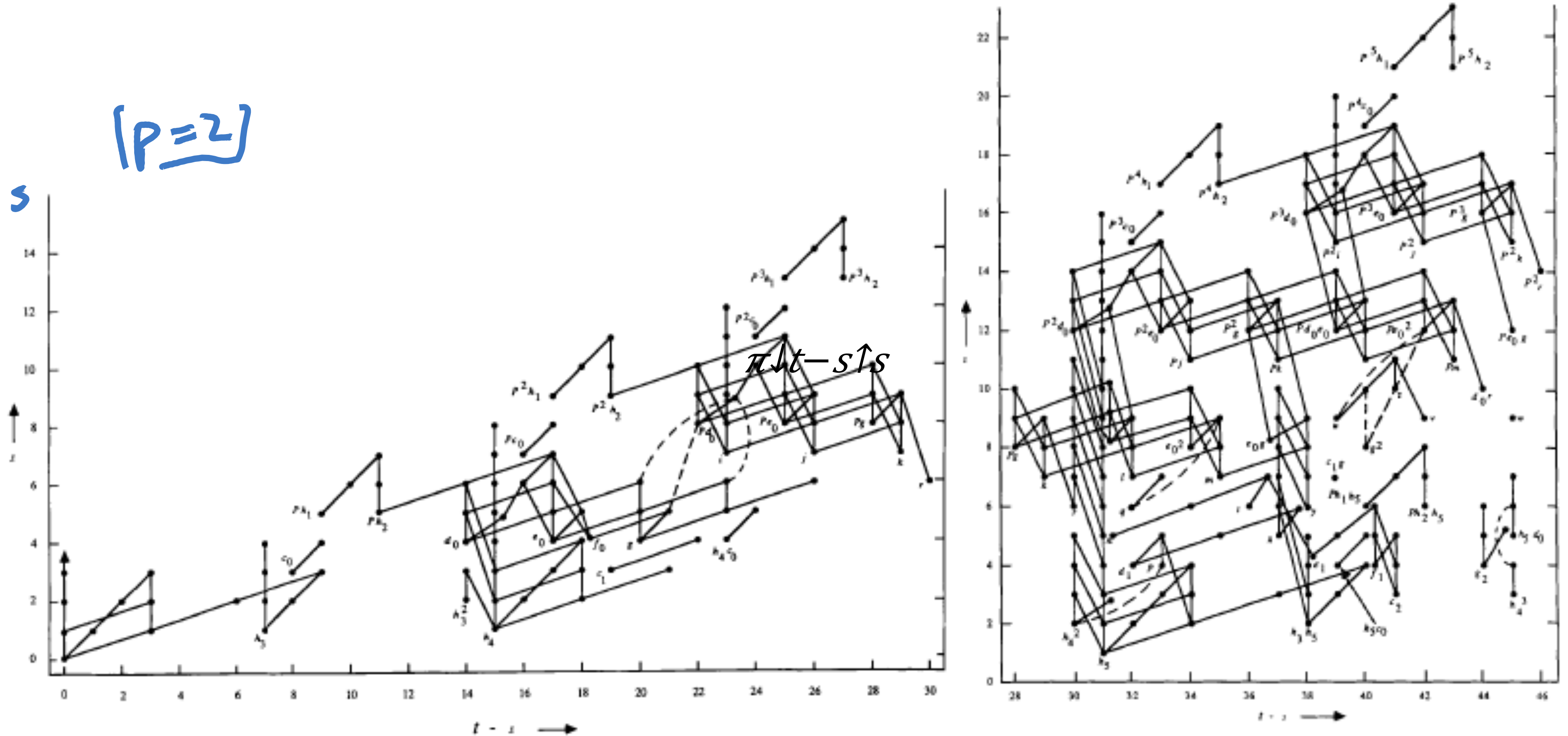


Figure A3.1a The Adams spectral sequence for $p = 2$, $t - s \leq 29$.

Adams spectral sequence

$Ext\downarrow A\uparrow s,t\ (\mathbb{Z}/p\ ,\mathbb{Z}/p)\Rightarrow(\pi\downarrow t-s\uparrow s\ )\downarrow p$

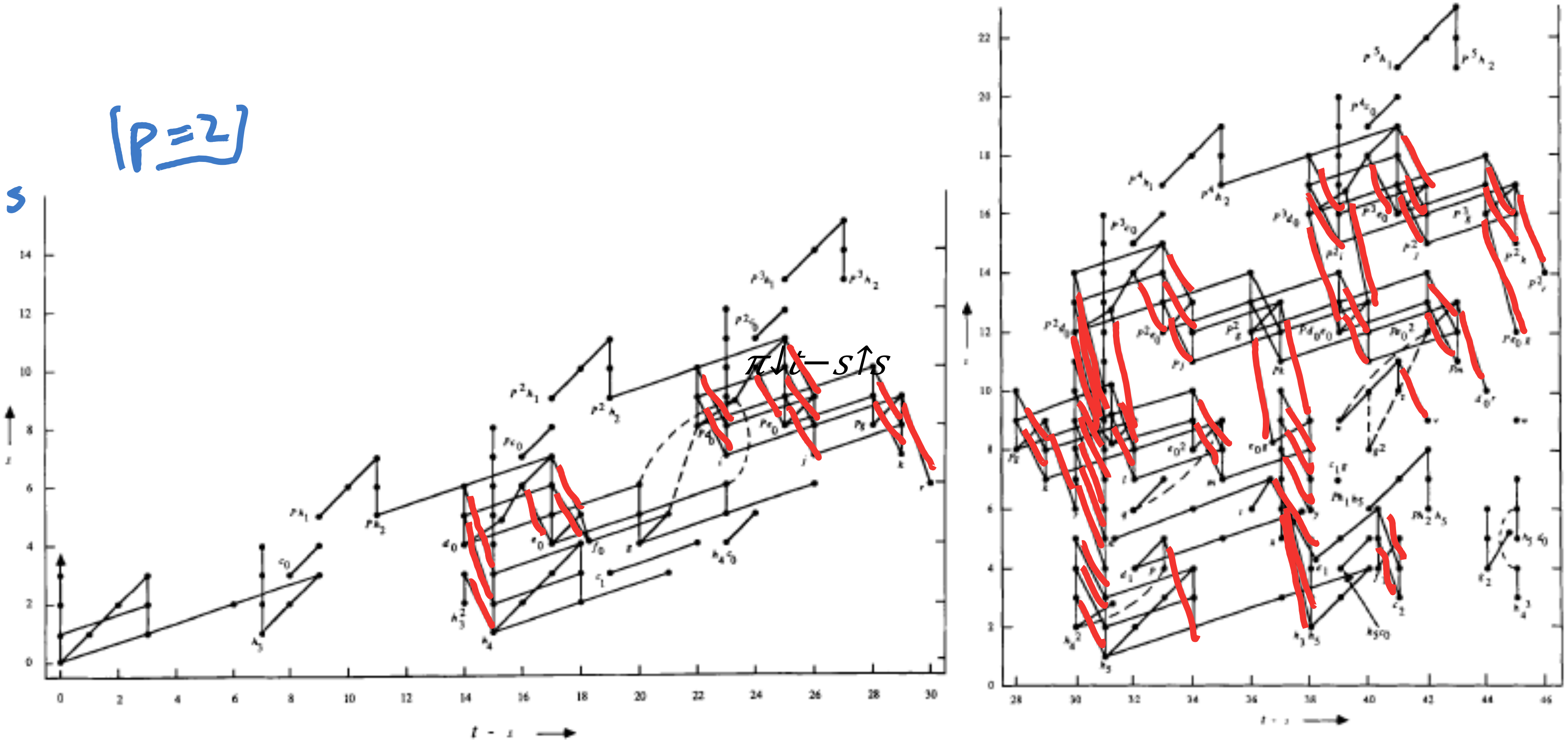


Figure A3.1a The Adams spectral sequence for $p = 2$, $t - s \leq 29$.

-Many differentials

-$d\downarrow r$ differentials go back by 1 and up by *r*

Adams spectral sequence

$\mathrm{Ext}_A^{s,t}(\mathbb{Z}/p,\mathbb{Z}/p) \Rightarrow (\pi_{t-s}^s)_p$

[p=2]

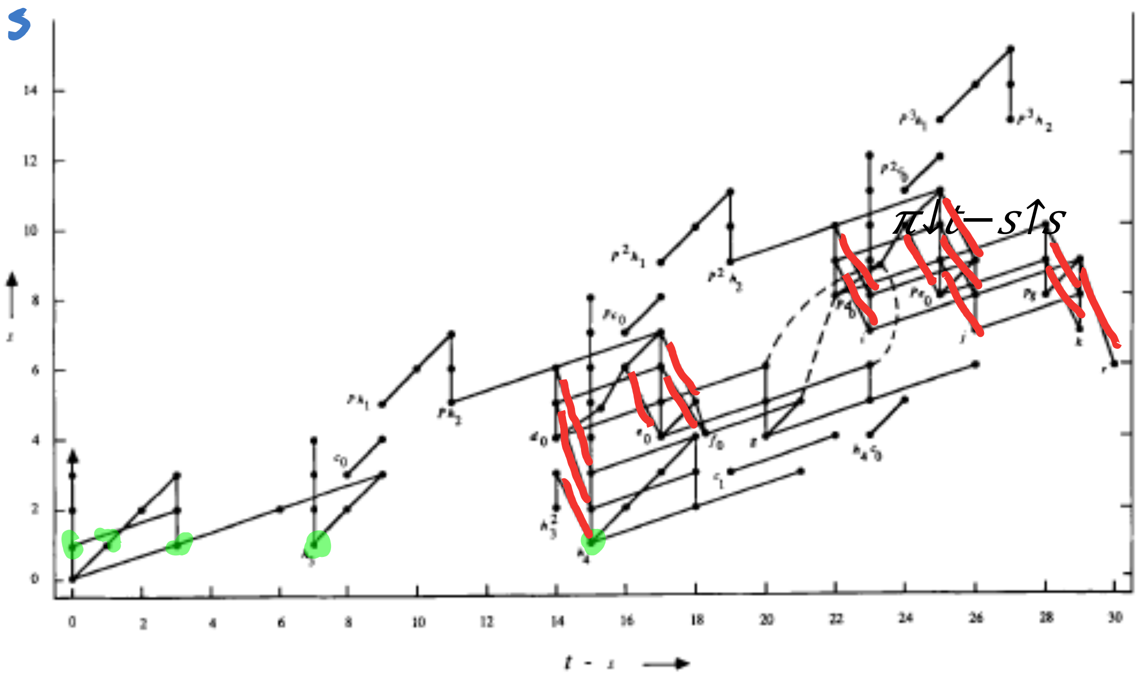
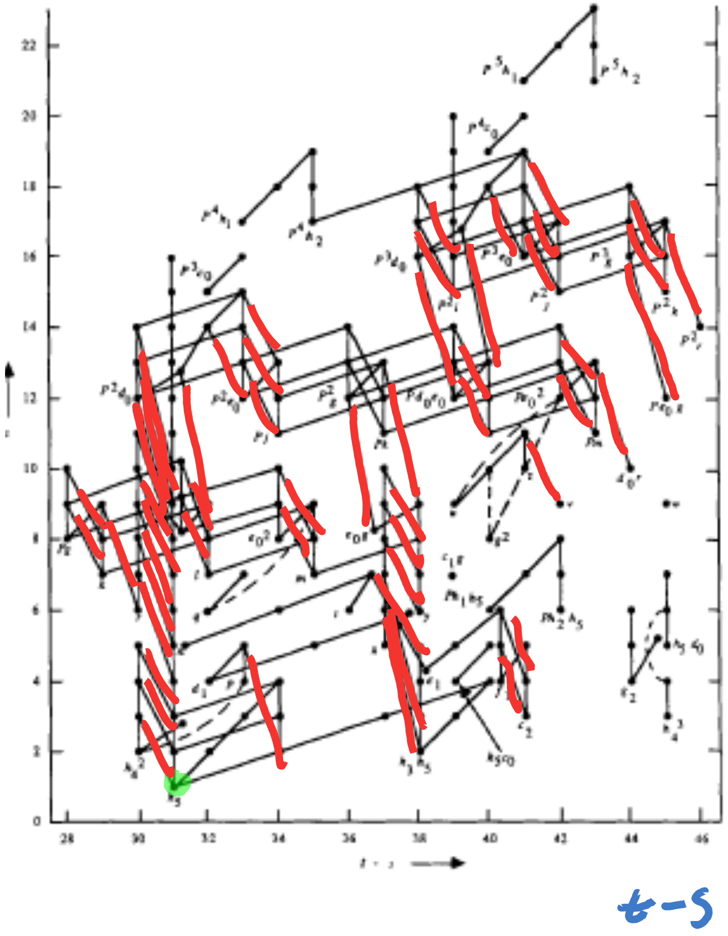


Figure A3.1a The Adams spectral sequence for $p = 2$, $t - s \leq 29$.

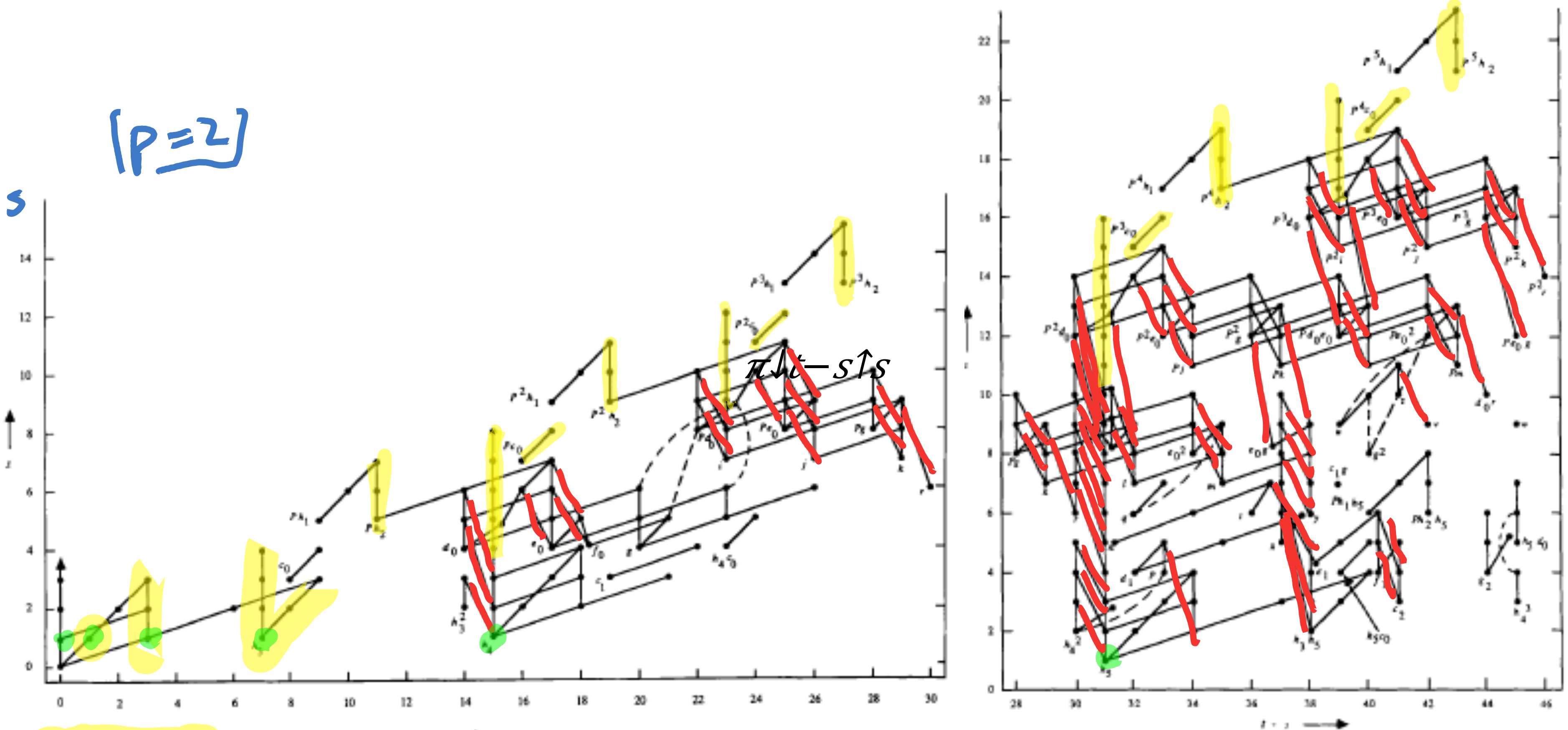


🟩: HI 1

- Many differentials
- d_r differentials go back by 1 and up by r

Adams spectral sequence

$Ext\downarrow A\uparrow s,t\,(\mathbb{Z}/p\,,\mathbb{Z}/p)\Rightarrow(\pi\downarrow t-s\uparrow s\,)\downarrow p$



🟩 : HI 1

Im J

Figure A3.1a The Adams spectral sequence for $p = 2$, $t - s \leq 29$.

Adams spectral sequence

$Ext\downarrow A\uparrow s,t\,(\mathbb{Z}/p\,,\mathbb{Z}/p)\Rightarrow(\pi\downarrow t-s\uparrow s\,)\downarrow p$

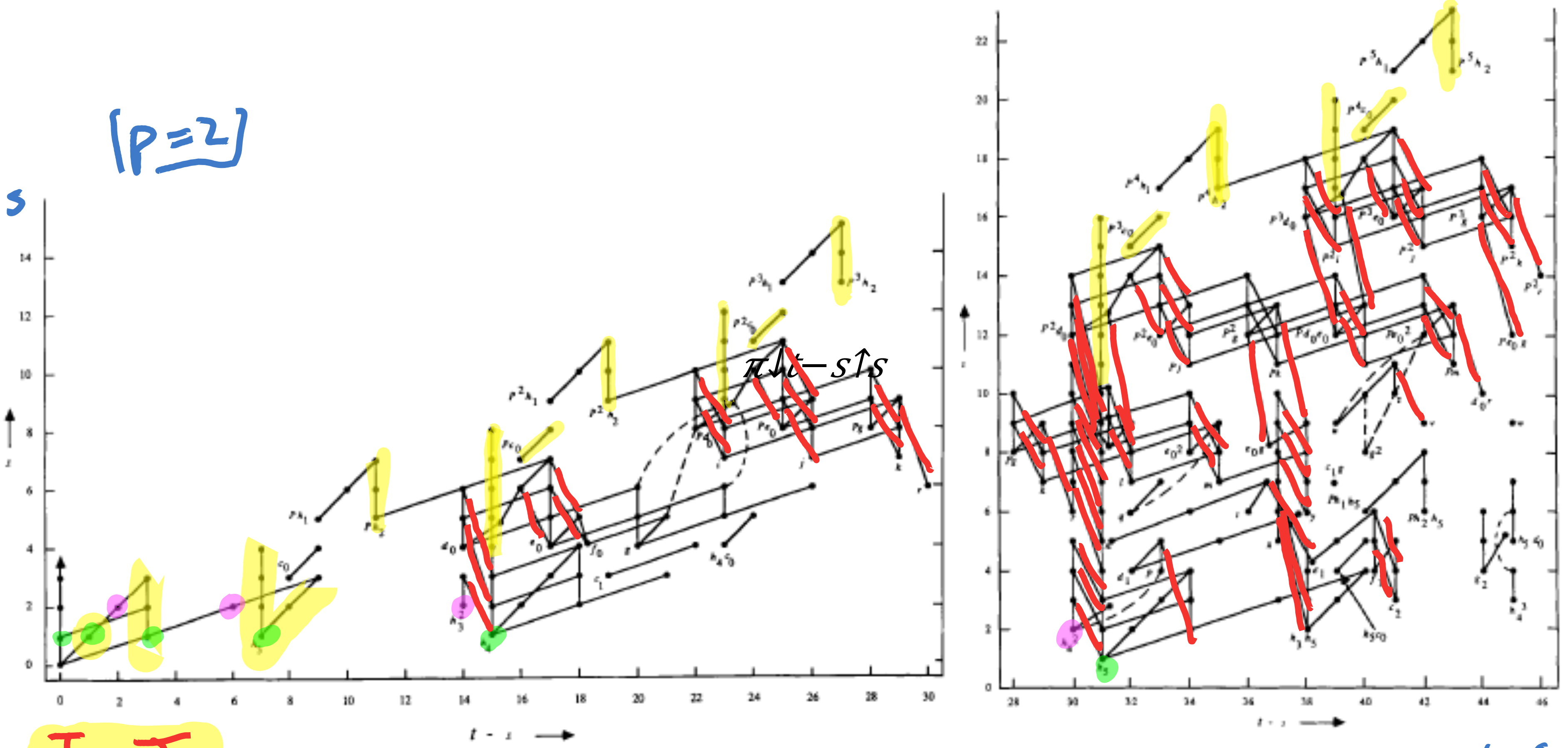


Figure A3.1a The Adams spectral sequence for $p = 2$, $t - s \leq 29$.

t−s

🟩 : HI 1 Im J

🟪 = Kervaire Invariant 1 (θ_j)

Adams spectral sequence

$Ext\downarrow A\uparrow s,t\,(\mathbb{Z}/p\,,\mathbb{Z}/p)\Rightarrow(\pi\downarrow t-s\uparrow s\,)\downarrow p$

[p=2]

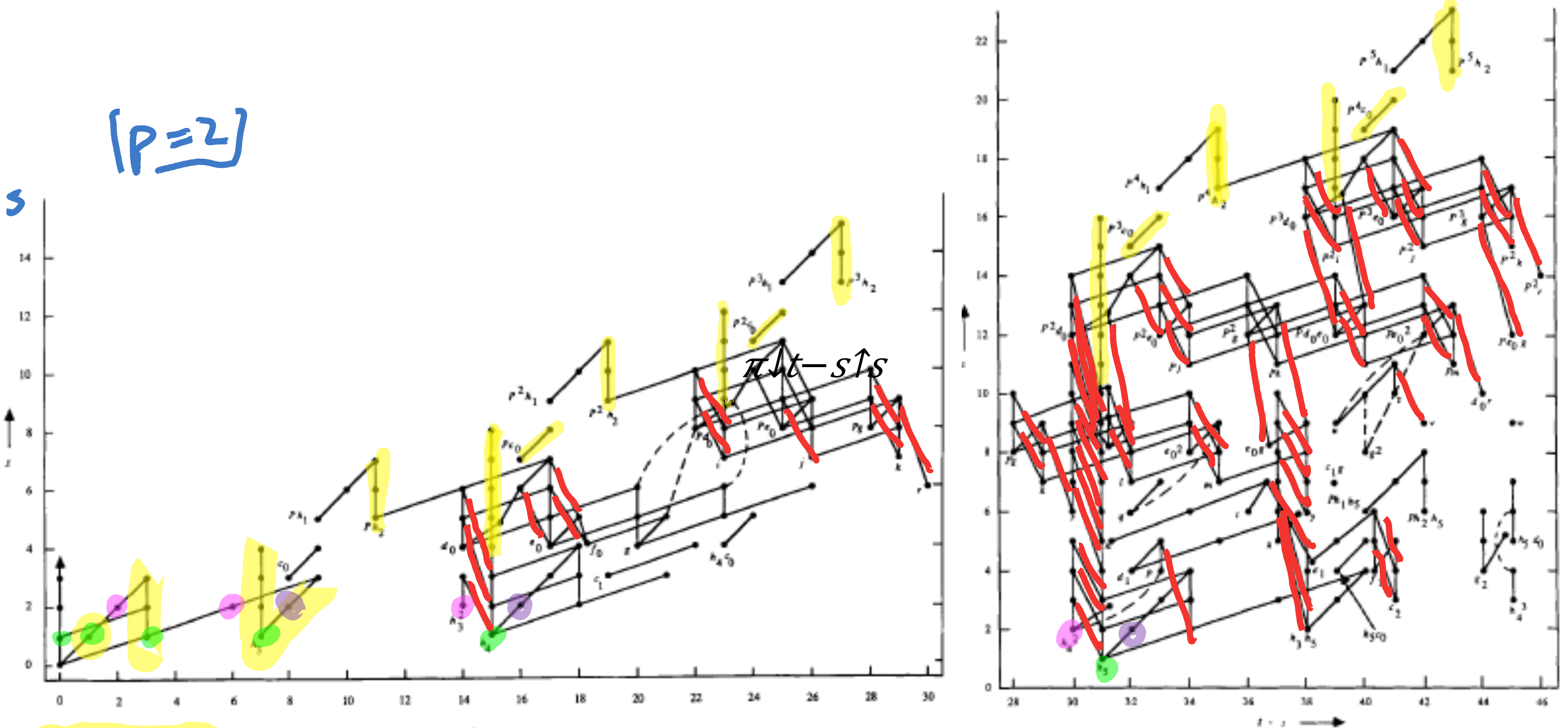


Figure A3.1a The Adams spectral sequence for $p = 2$, $t - s \leq 29$.

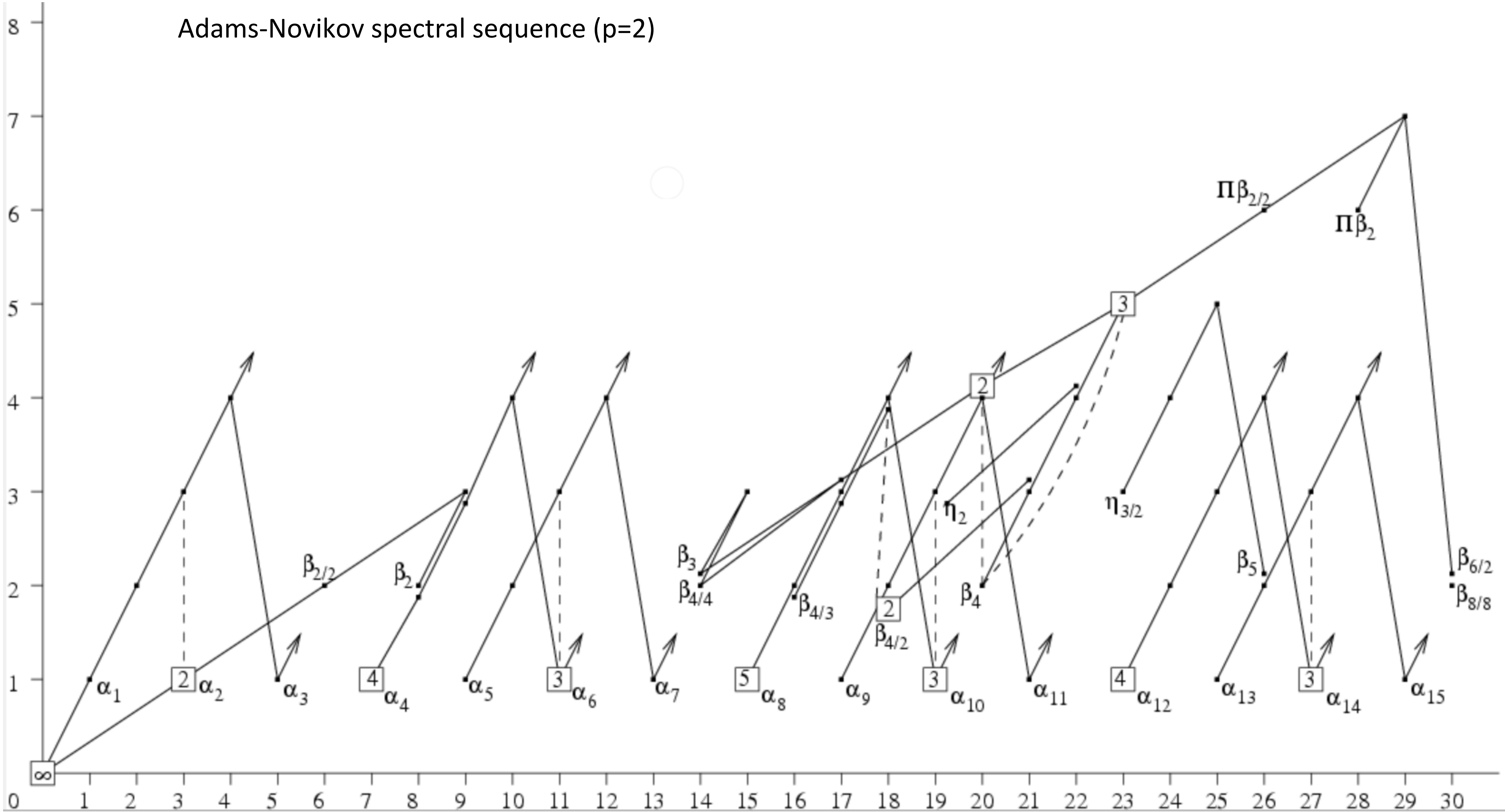
t−s

🟩 : HI 1 Im J

🟪 = Ker Invariant 1 (θ_j)

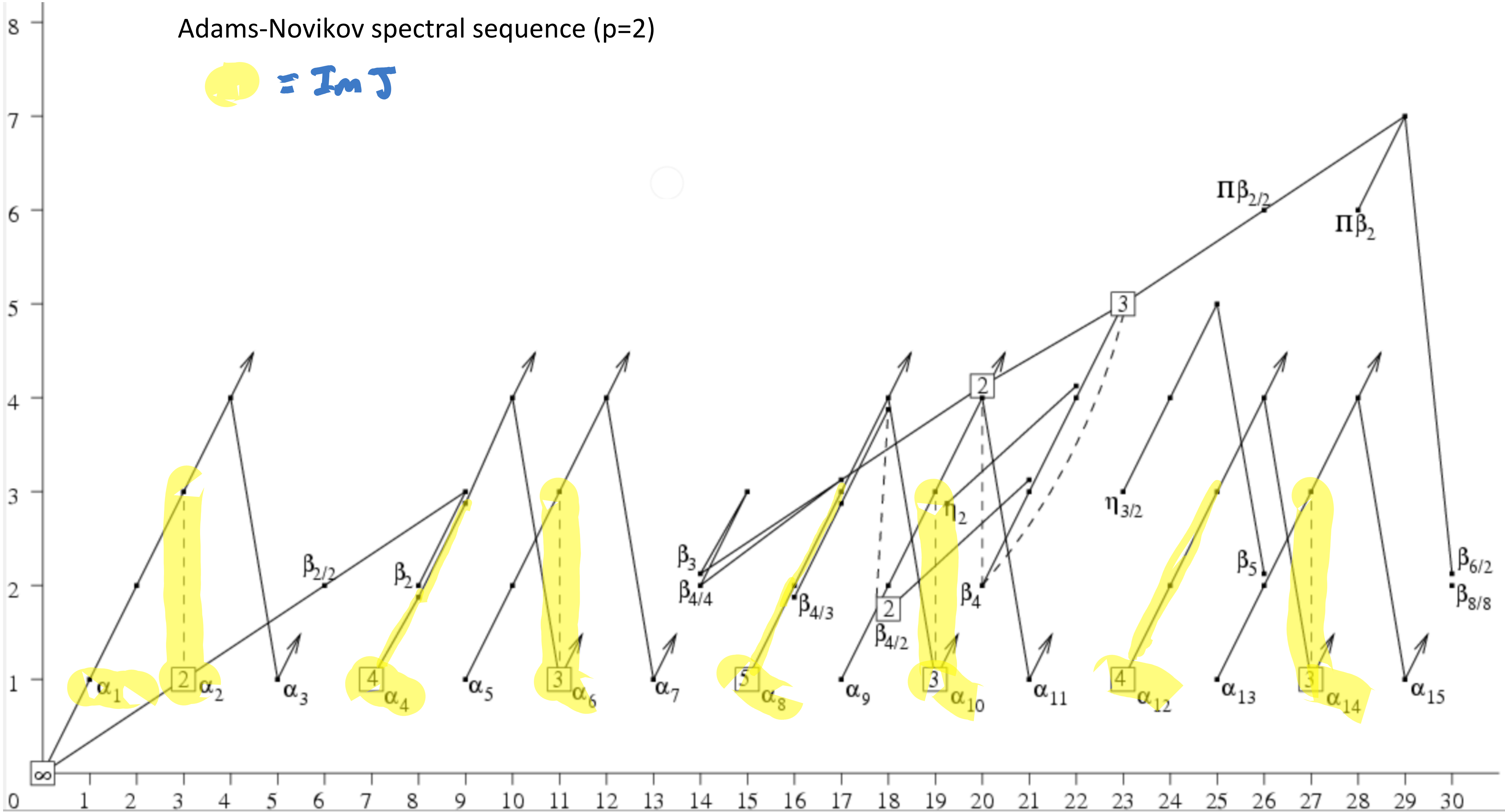
🟪 : Mahowald η_j family

Adams-Novikov spectral sequence (p=2)



Adams-Novikov spectral sequence (p=2)

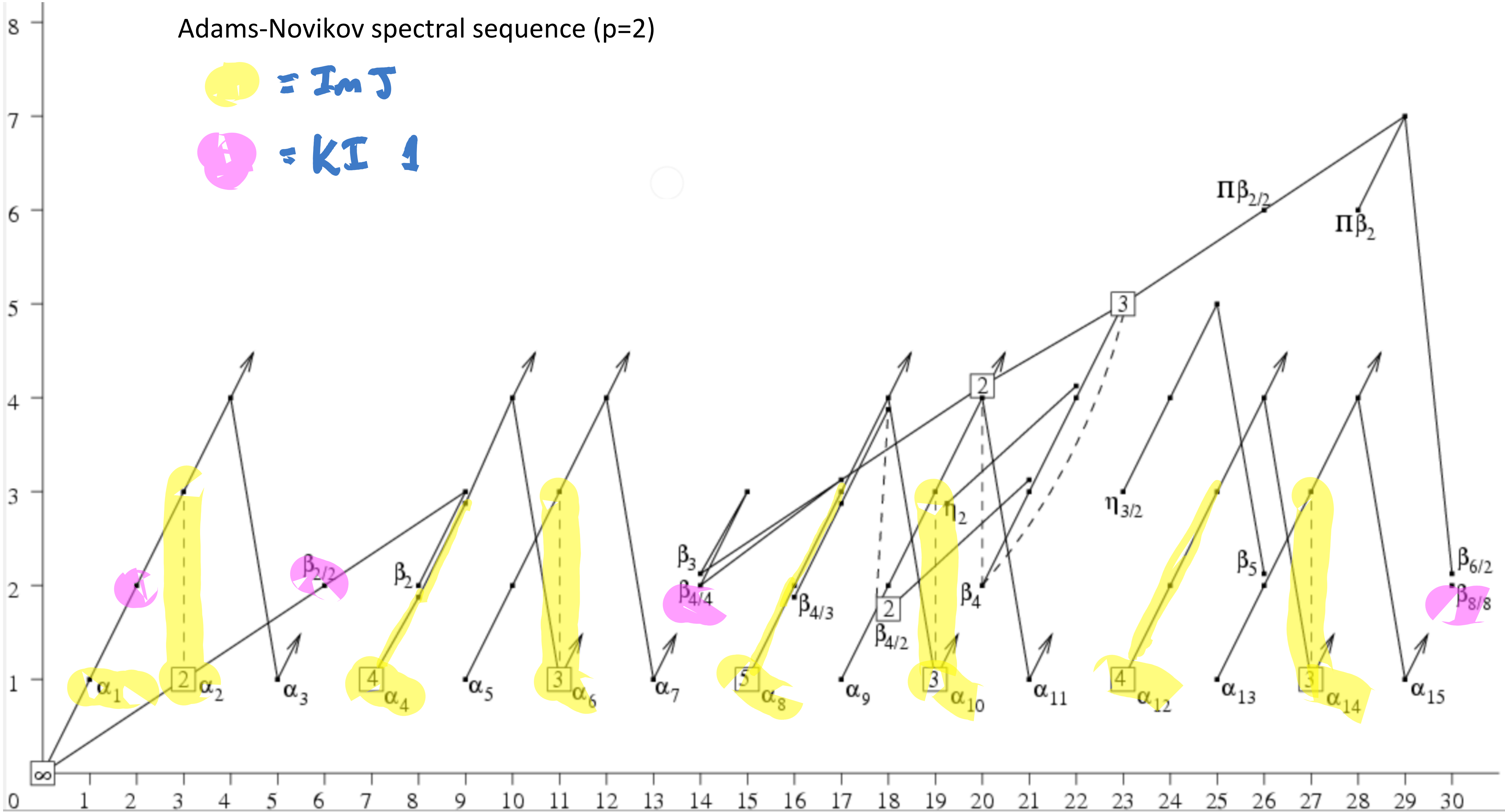
🟡 = Im J



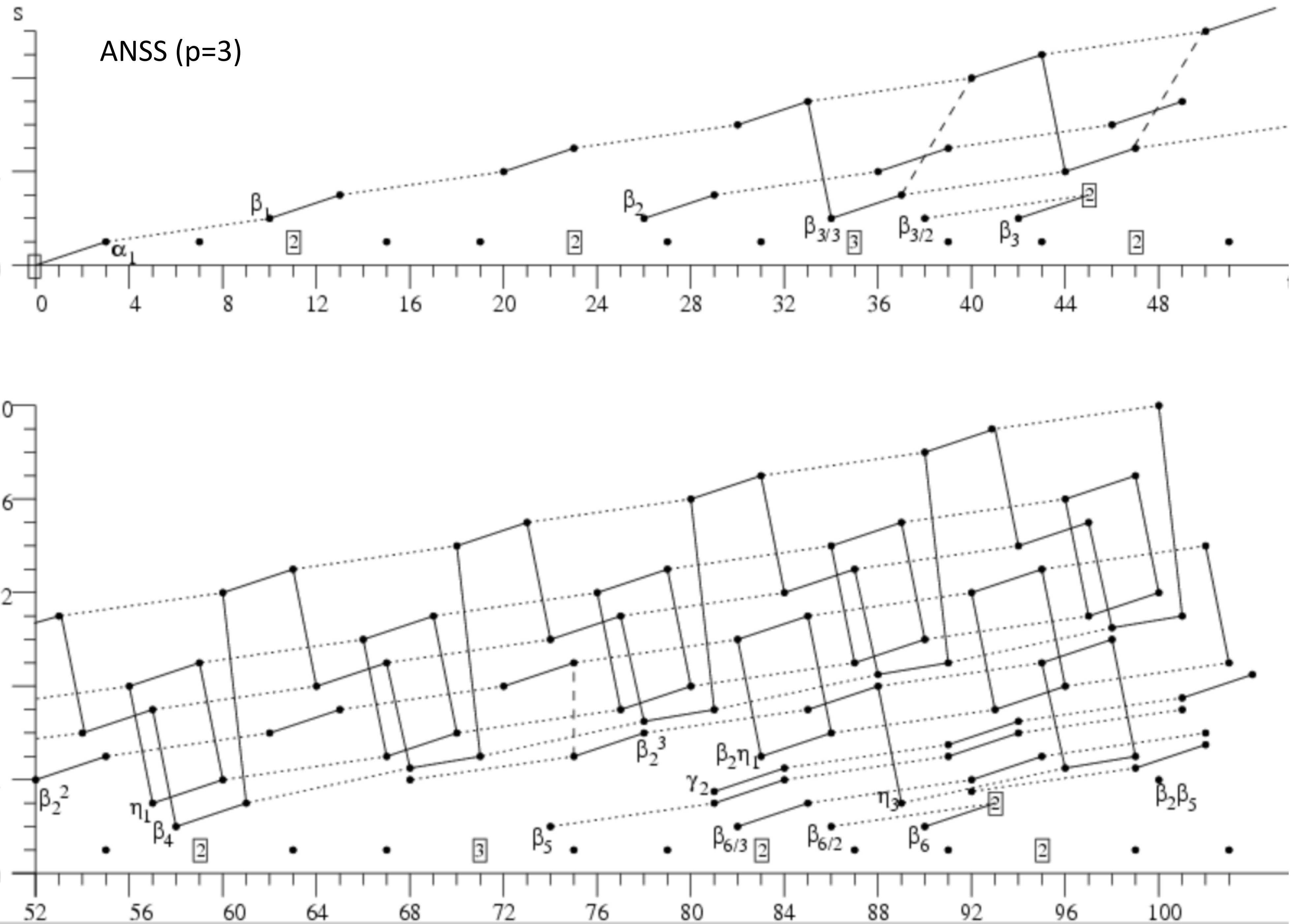
Adams-Novikov spectral sequence (p=2)

🟡 = Im J

🟣 = KI 1

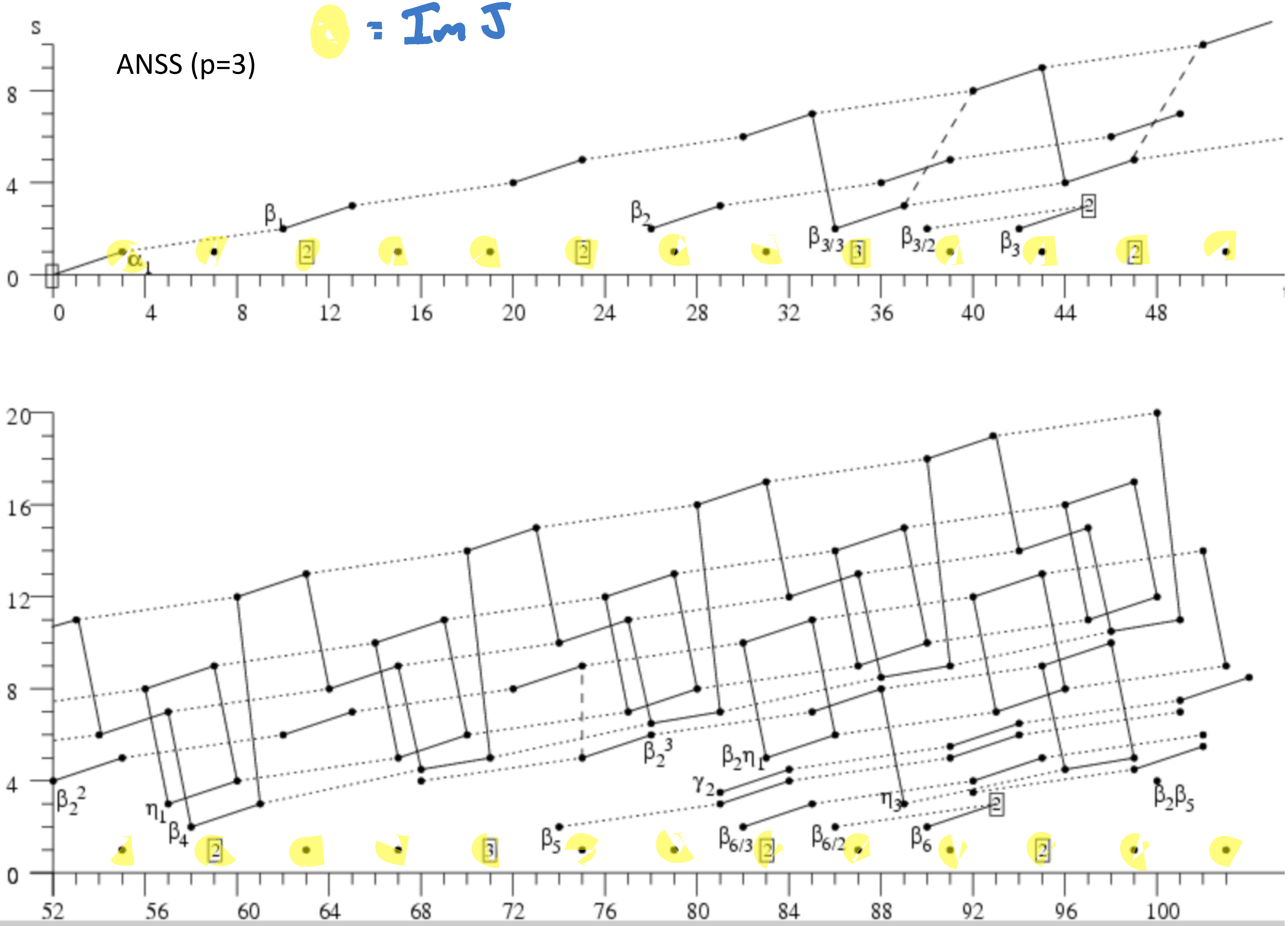


ANSS (p=3)


s
0
4
8
12
16
20
24
28
32
36
40
44
48
52
56
60
64
68
72
76
80
84
88
92
96
100
α_1
β_1
β_2
$\beta_{3/3}$
$\beta_{3/2}$
β_3
2
3
β_2^2
η_1
β_4
β_5
β_2^3
$\beta_2\eta_1$
γ_2
$\beta_{6/3}$
$\beta_{6/2}$
η_3
β_6
$\beta_2\beta_5$

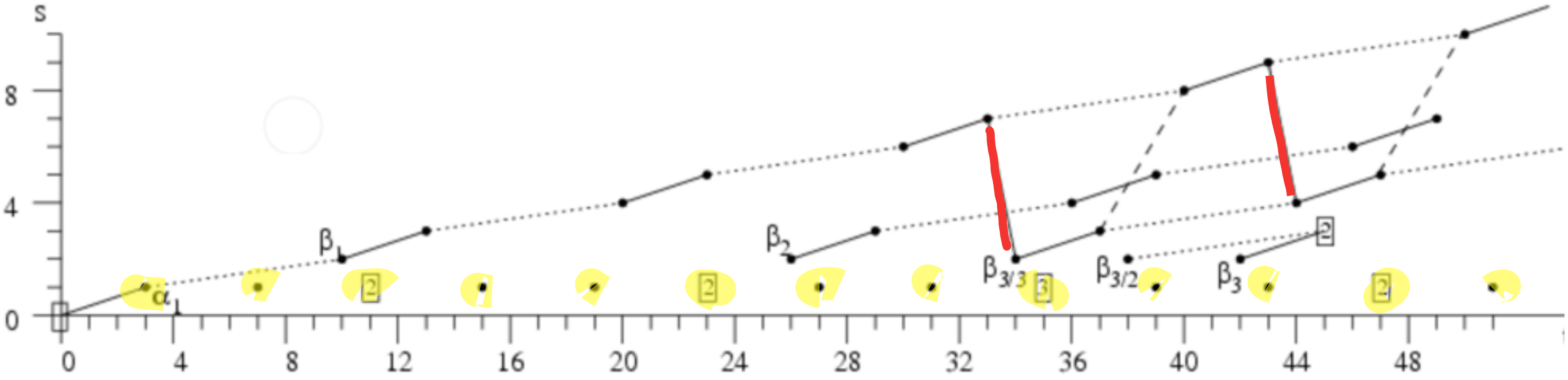
ANSS (p=3)

● = Im J


s
0
4
8
0
4
8
12
16
20
24
28
32
36
40
44
48
α1
β1
β2
β3/3
β3/2
β3
2
3
2
2
2
2
20
16
12
8
4
0
52
56
60
64
68
72
76
80
84
88
92
96
100
β2^2
η1
β4
β5
β2^3
β2η1
γ2
β6/3
β6/2
η3
β6
β2β5
2
3
2
2
2
2

ANSS (p=3)

🟨 = Im J



s
8
4
0
α_1
β_1
β_2
$\beta_{3/3}$
$\beta_{3/2}$
β_3
2
3
0
4
8
12
16
20
24
28
32
36
40
44
48

ASS (p=3)

🟨 = Im J

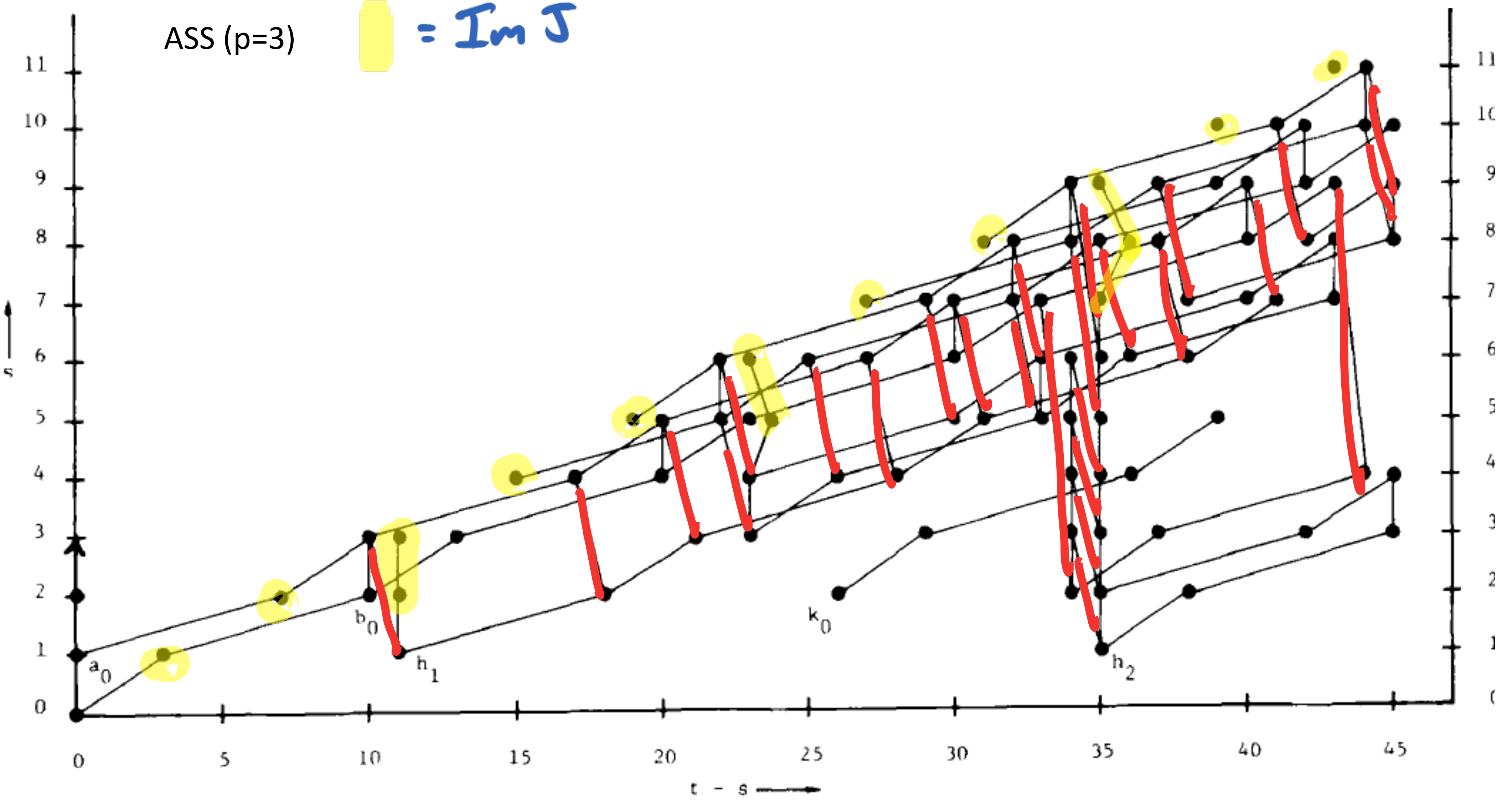
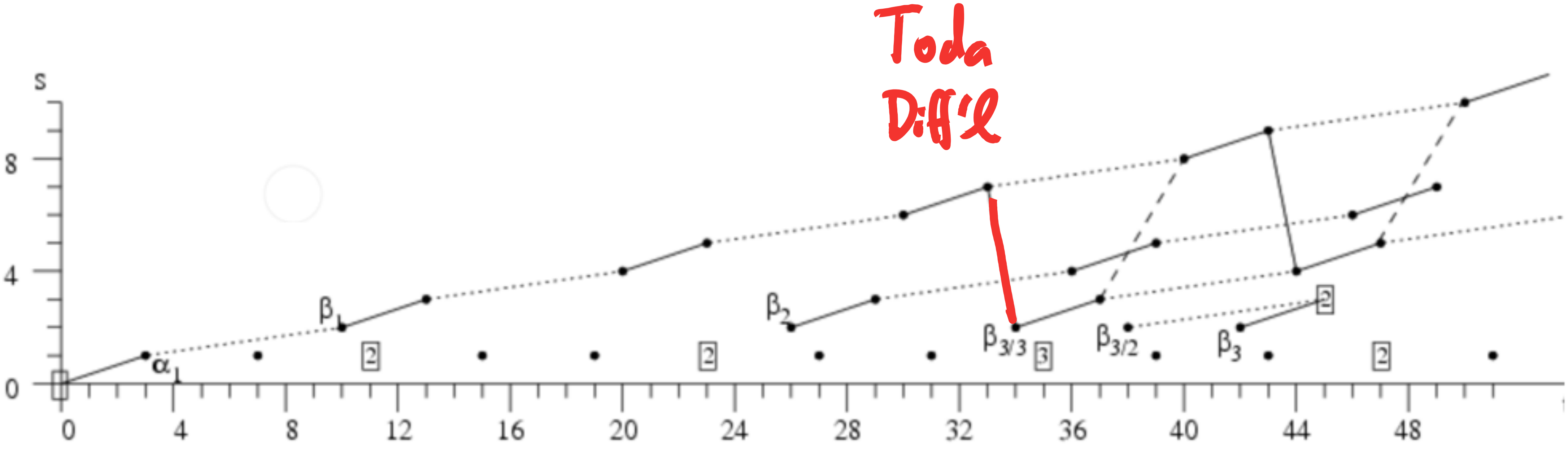
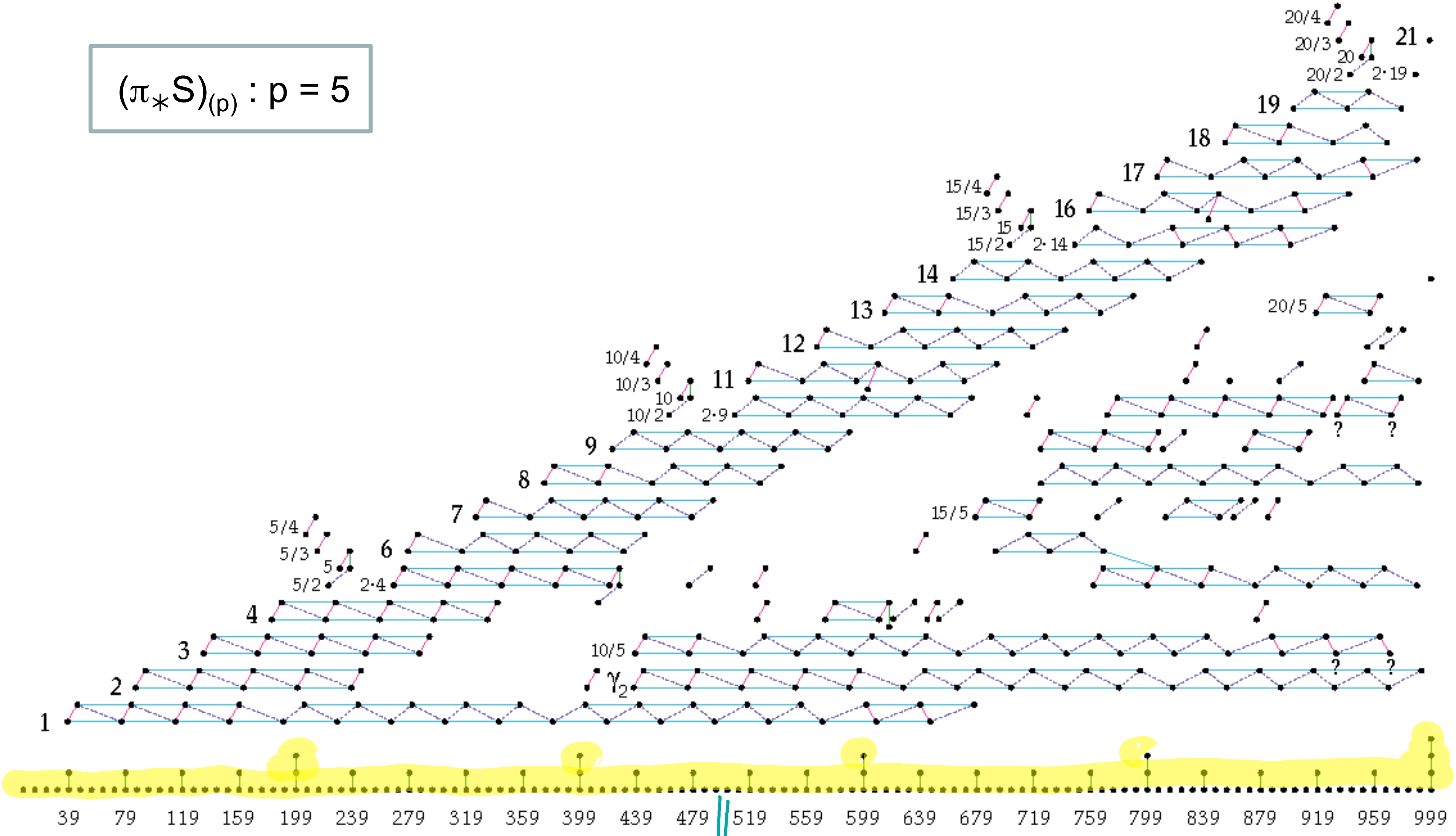


Figure 1.2.15 The Adams spectra sequence for $p = 3$, $t - s \leq 45$.

ANSS (p=3)


Toda
Diff'l
s
8
4
0
α1
β1
β2
β3/3
β3/2
β3
2
3
0
4
8
12
16
20
24
28
32
36
40
44
48

$(\pi_*S)_{(p)} : p = 5$

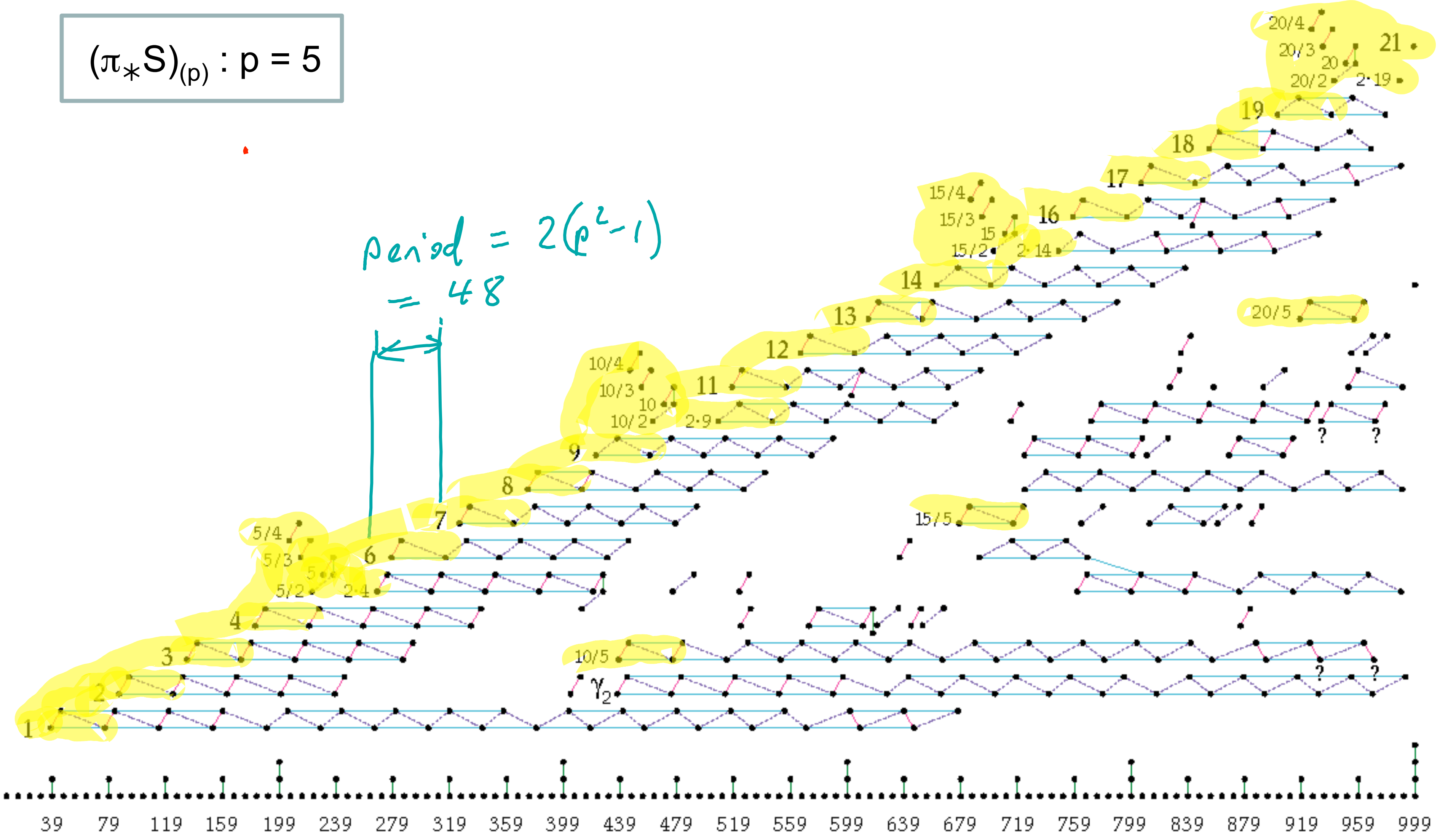


period $= 2(p-1) = 8$

v_1-periodic $= \mathrm{Im}J$

Computation: Ravenel
Picture: Hatcher

$(\pi_* S)_{(p)}$: p = 5



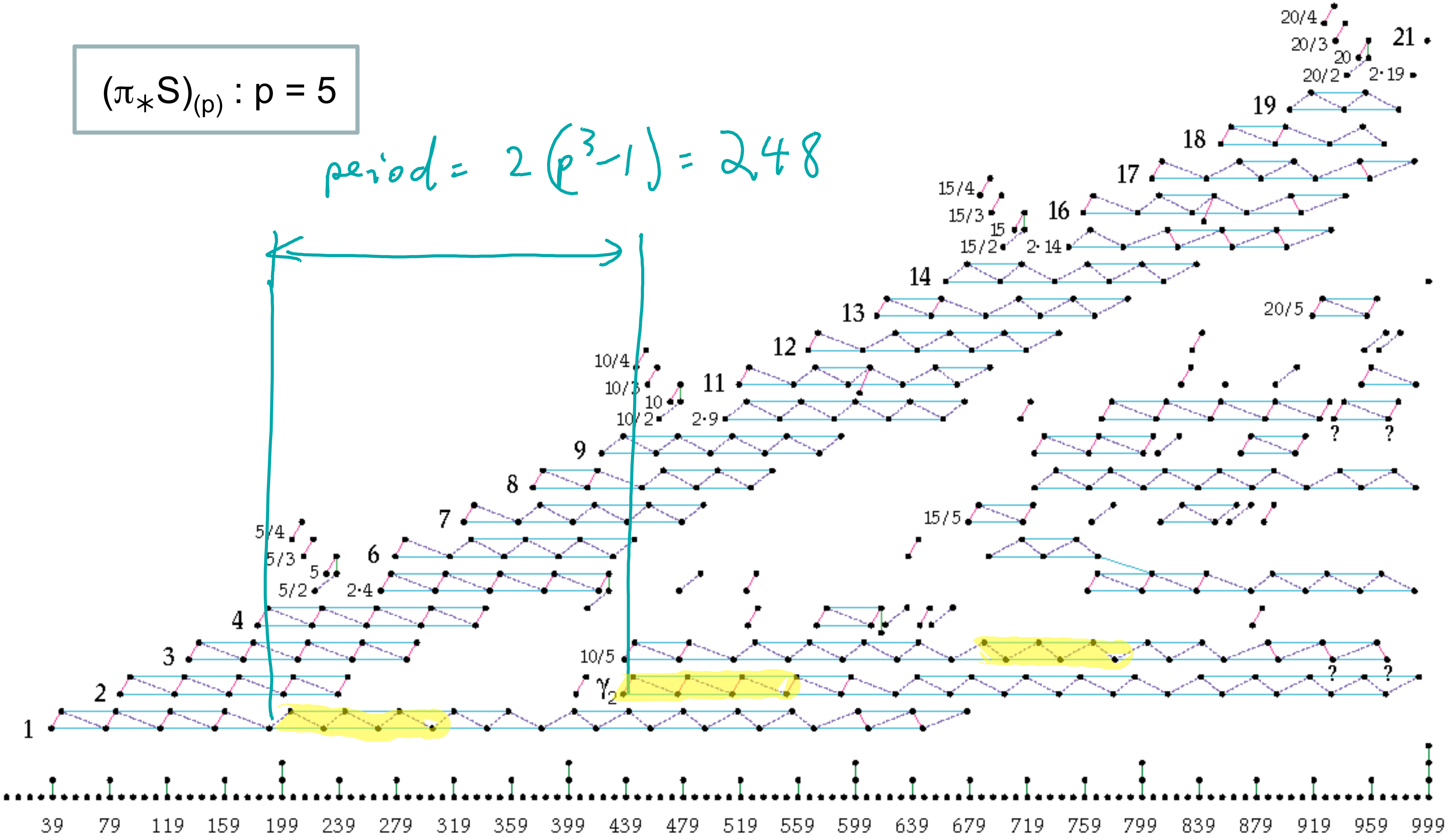
period = $2(p^2-1)$
= 48

v_2 - periodic

Computation: Ravenel
Picture: Hatcher

$(\pi_* S)_{(p)} : p = 5$

period $= 2(p^3-1) = 248$



v_3-periodic

Computation: Ravenel
Picture: Hatcher

Greek letter elements

The most fundamental v_n-periodic elts are the GREEK LETTER ELTS

Greek letter elements

The most fundamental v_n-periodic elts are the GREEK LETTER ELTS

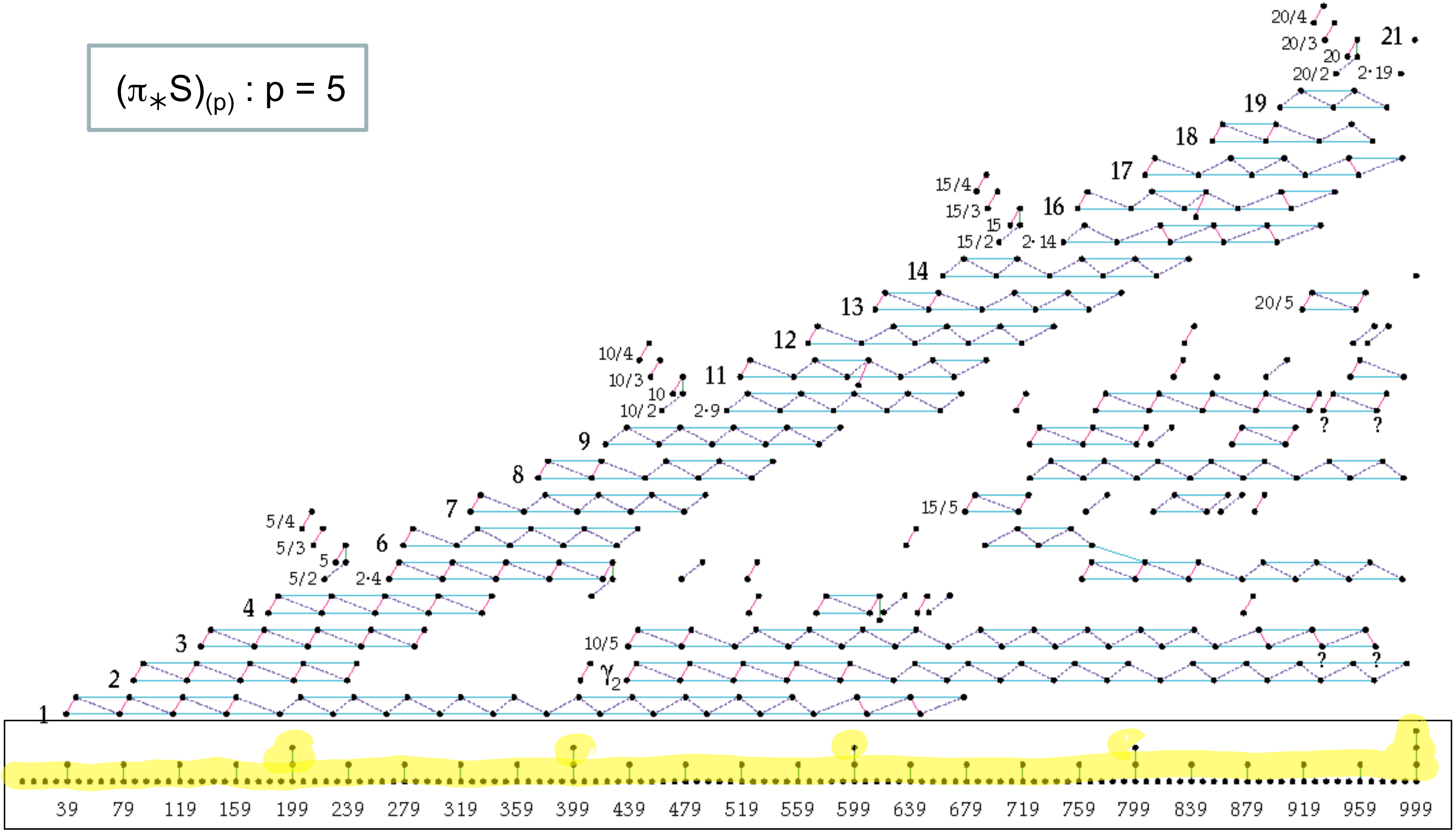
Notation

v_1-periodic: $\alpha_{i/j}$

v_2-periodic: $\beta_{i/j,k}$

v_3-periodic: $\gamma_{i/j,k,l}$

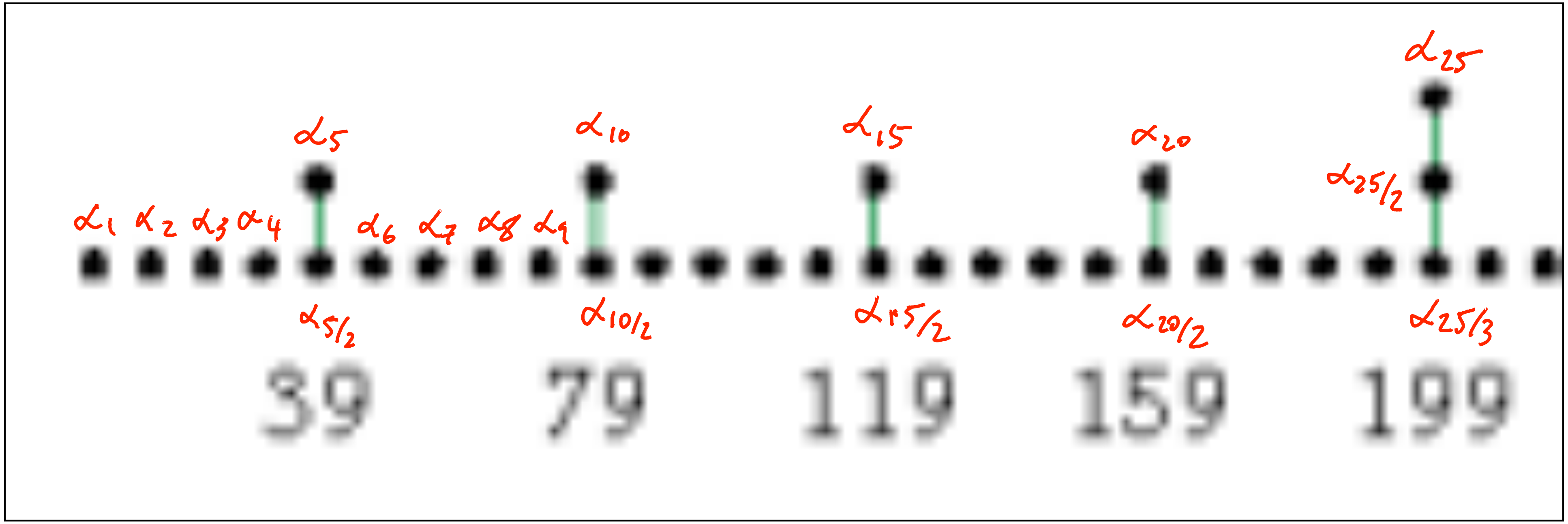
$(\pi_* S)_{(p)} : p = 5$



v_1-periodic: α-family

Greek letter notation:

$\alpha_{i/j} \in \left(\pi^{S}_{2(p-1)i-1}\right)$

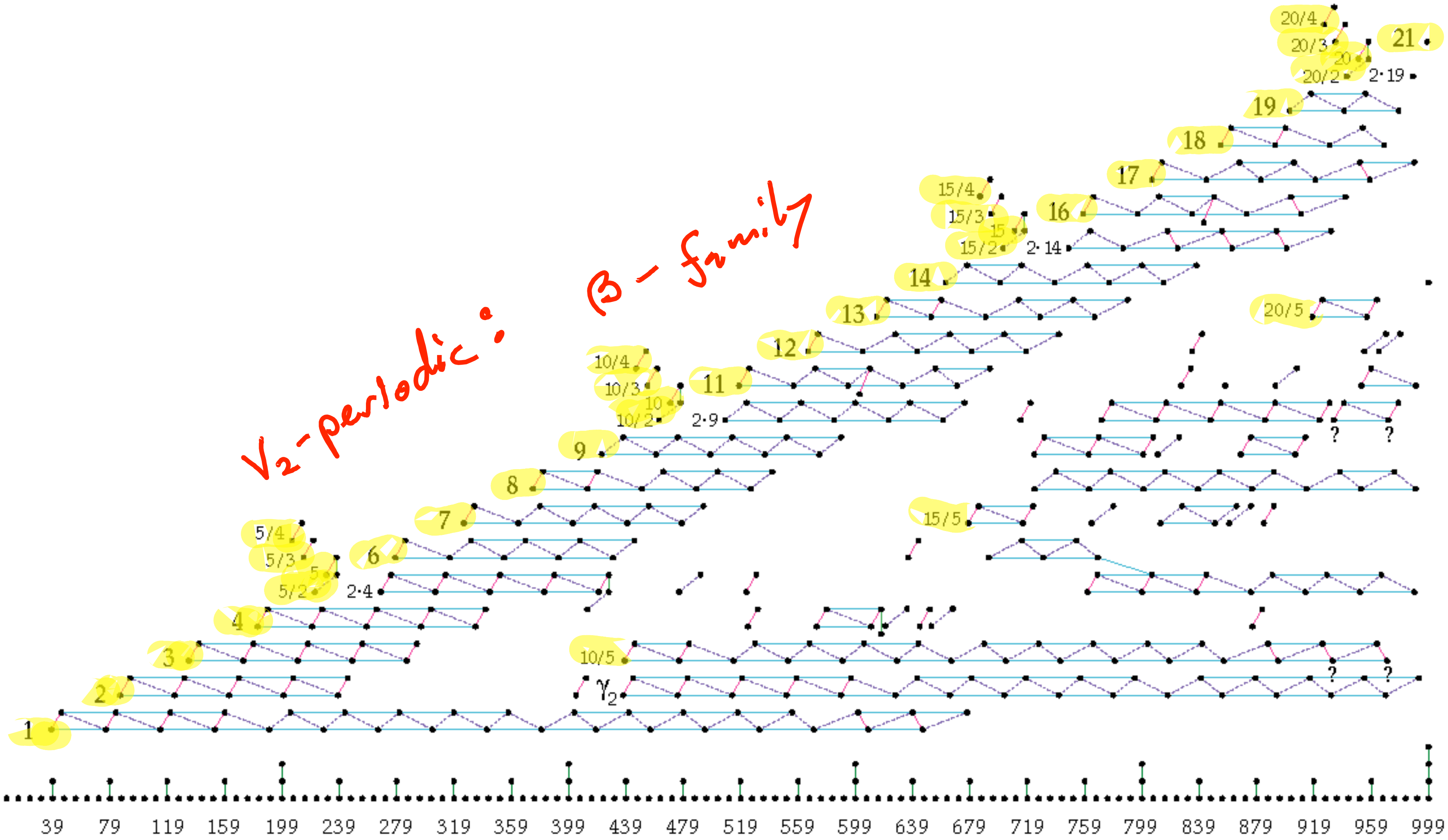


$\alpha_{i/j}$ is p^j - torsion

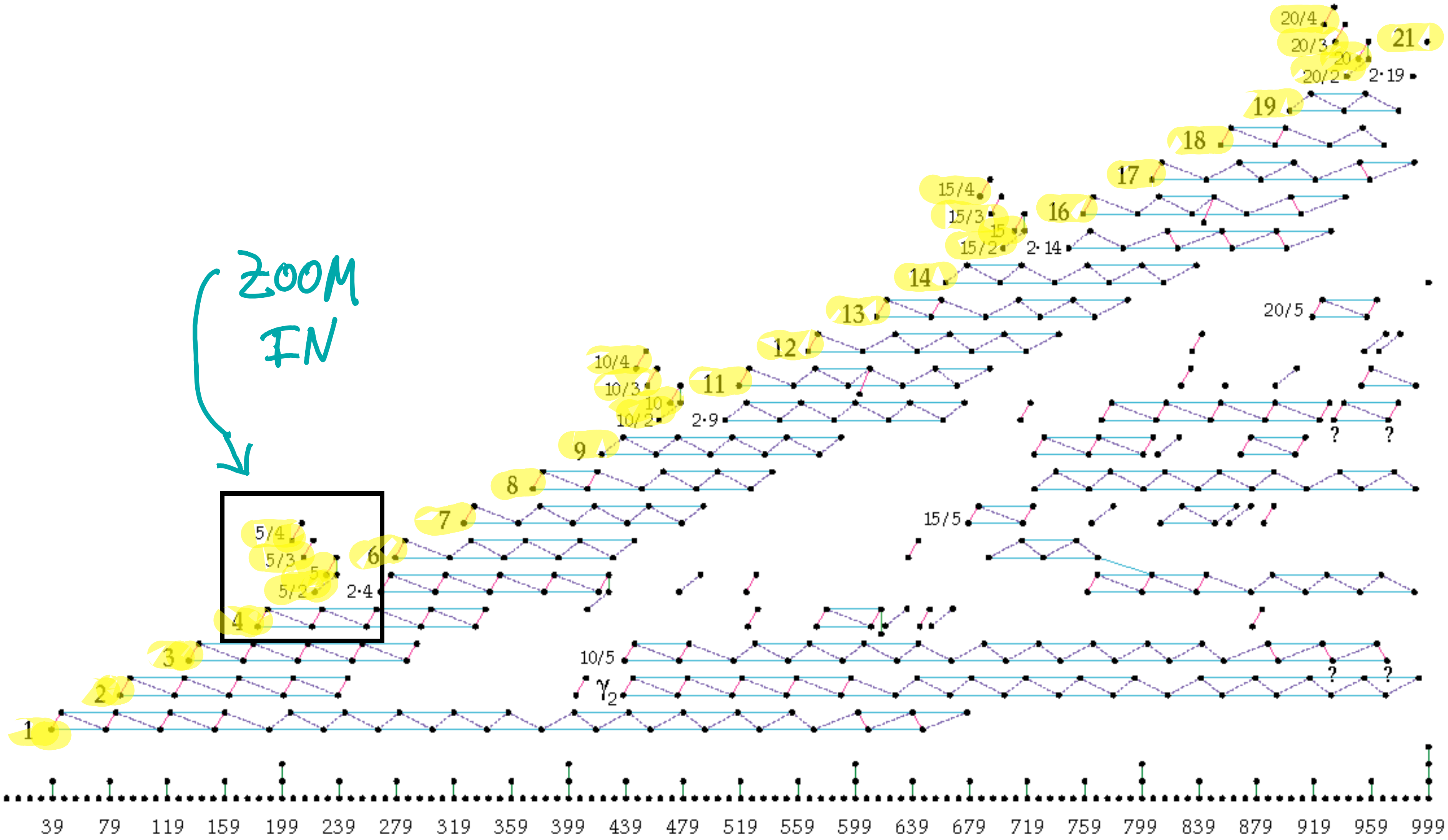
"ImJ pattern"

$(\alpha_i := \alpha_{i/1})$

$(\pi_* S)_{(p)}$: p = 5

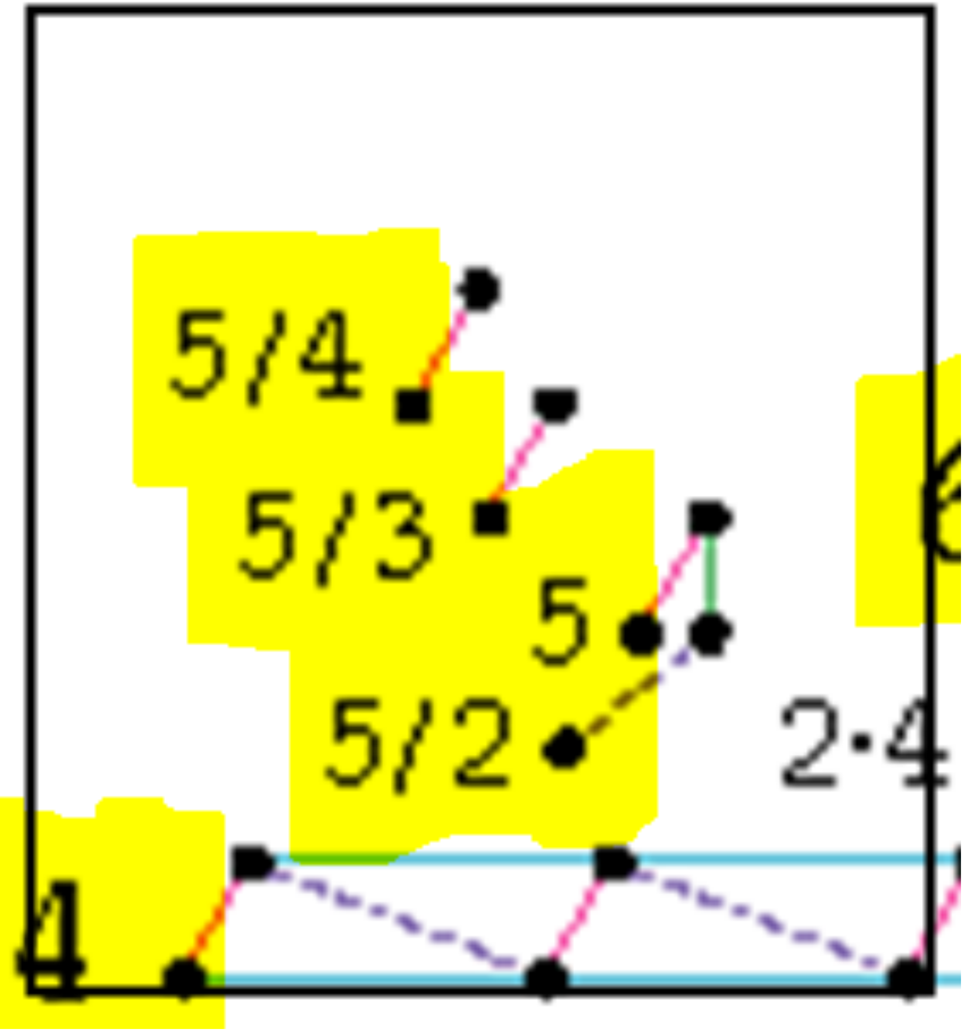

v_2-periodic : β - family

$(\pi_* S)_{(p)}$: p = 5



ZOOM IN

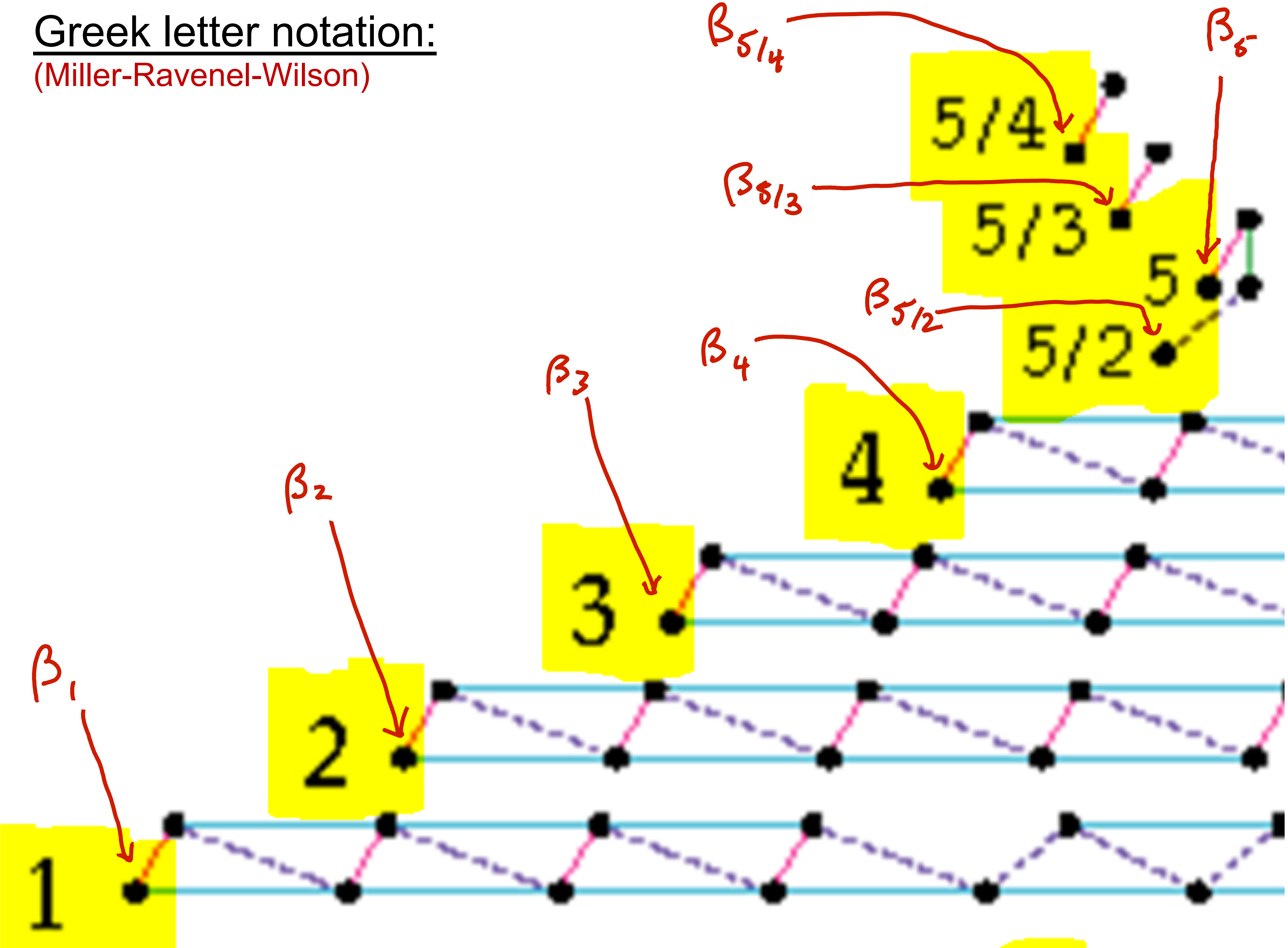
v_1 – torsion in the v_2 – family:



$$5/4 \xrightarrow[v_1]{} 5/3 \xrightarrow[v_1]{} 5/2 \xrightarrow[v_1]{} 5 \xrightarrow[v_1]{} 0$$

Greek letter notation:

(Miller-Ravenel-Wilson)



β-family notation:

$$\beta_{i/j,k} \in \left(\pi^s_{2(p^2-1)i-2(p-1)j-2}\right)_{(p)}$$

p^k-torsion

$$v_2 \beta_{i/j,k} = \beta_{i+1/j,k}$$

$$v_1 \beta_{i/j,k} = \beta_{i/j-1,k}$$

$$p \beta_{i/j,k} = \beta_{i/j,k-1}$$

β-family notation:

$$\beta_{i/j,k} \in \left(\pi^s_{2(p^2-1)i-2(p-1)j-2}\right)_{(p)}$$

p^k-torsion

Convention

$$\beta_{i/j} := \beta_{i/j,1}$$

$$\beta_i := \beta_{i/1}$$

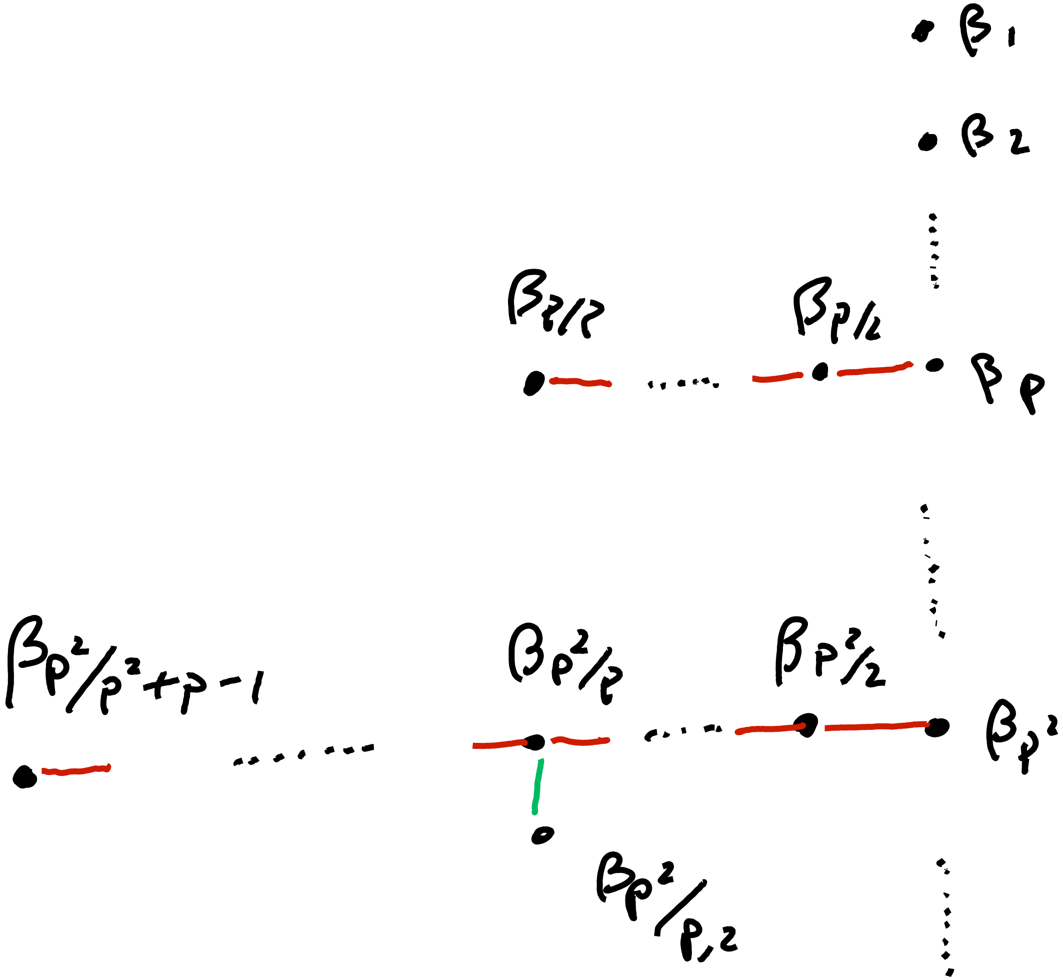
$$v_2 \beta_{i/j,k} = \beta_{i+1/j,k}$$

$$v_1 \beta_{i/j,k} = \beta_{i/j-1,k}$$

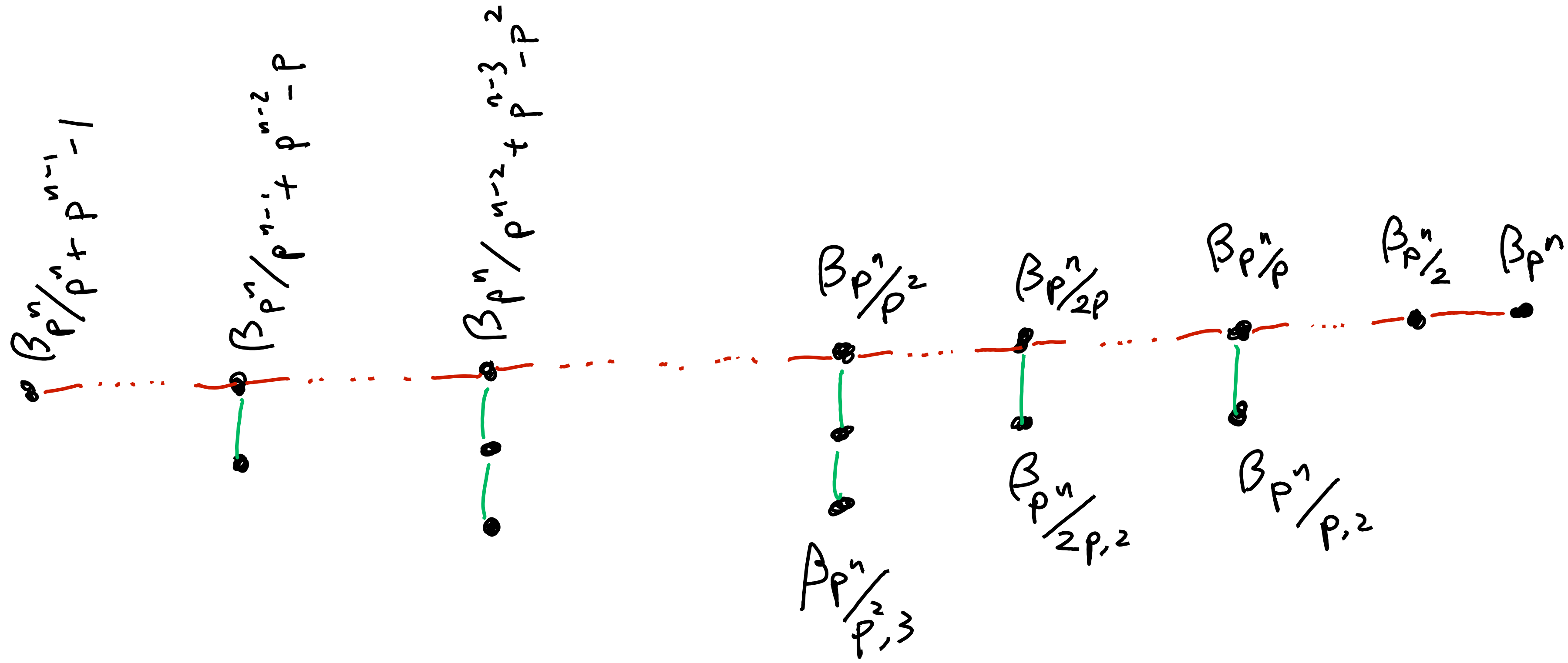
$$p \beta_{i/j,k} = \beta_{i/j,k-1}$$

Description of the β-family:

(Miller-Ravenel-Wilson ‘77)

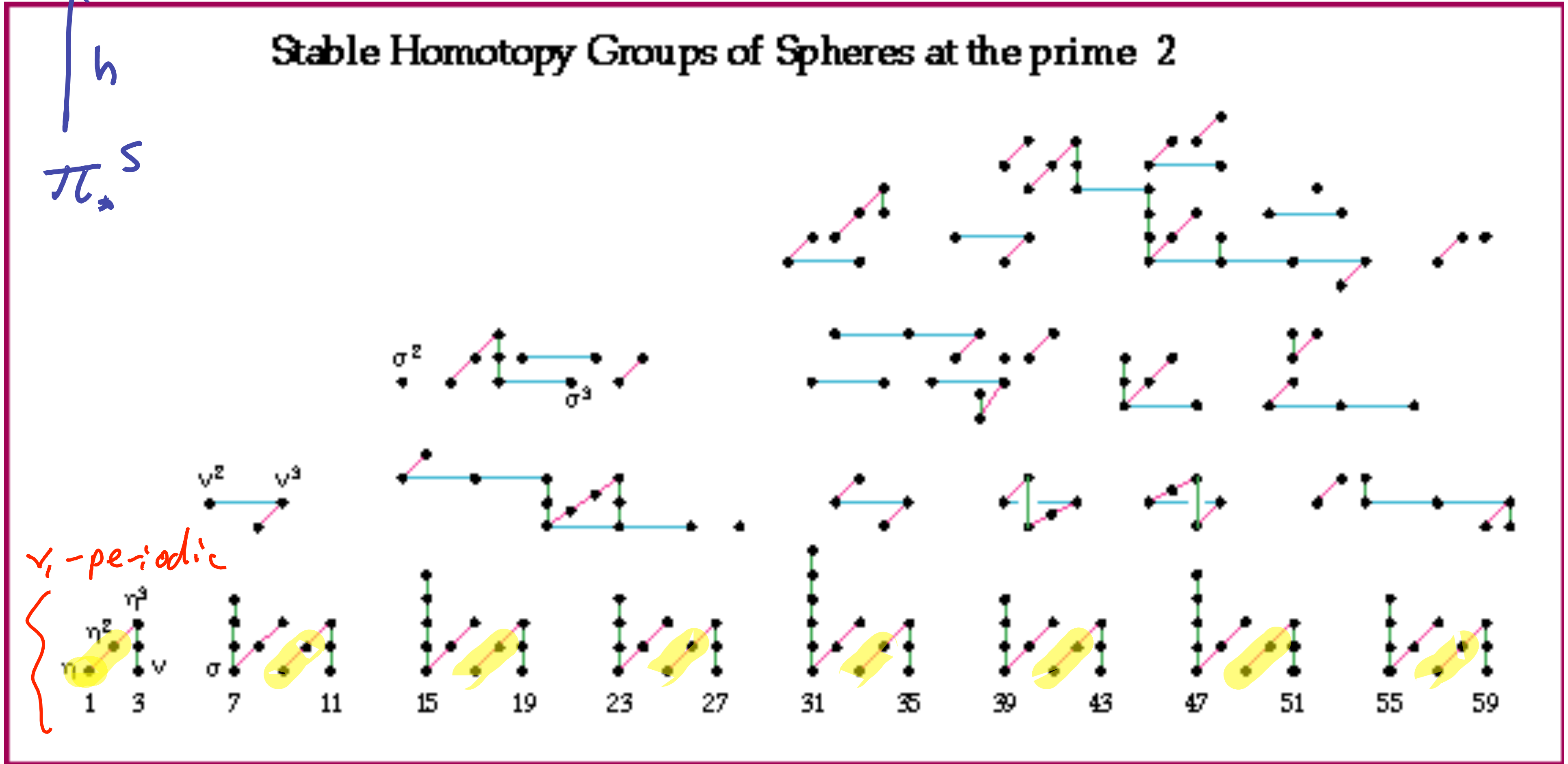


Description of the β-family:



KO Hurewicz homomorphism

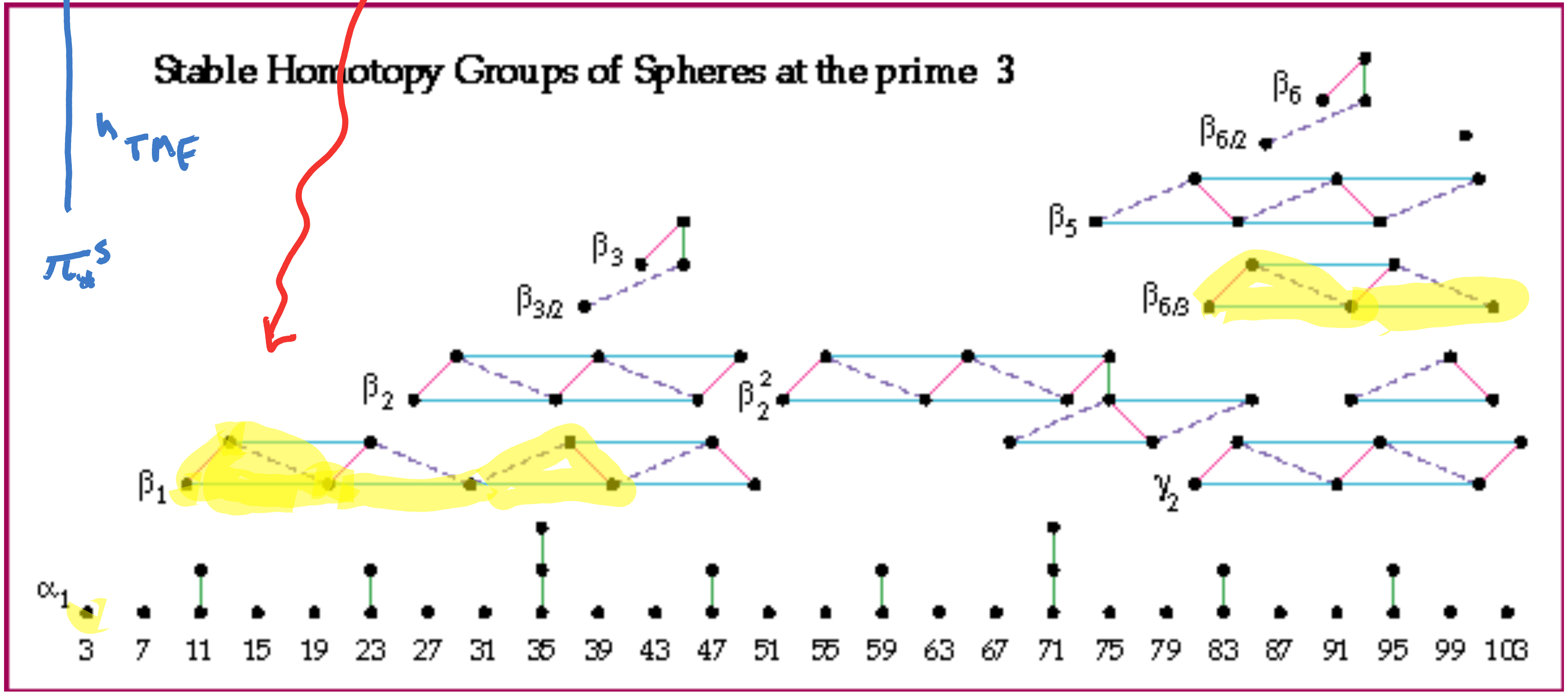
$$\begin{tikzcd} \pi_* KO = \mathbb{Z} \quad \mathbb{Z}/2 \quad \mathbb{Z}/2 \quad 0 \quad \mathbb{Z} \quad 0 \quad 0 \quad 0 \quad \mathbb{Z} \quad \mathbb{Z}/2 \quad \mathbb{Z}/2 \quad 0 \quad \mathbb{Z} \dots \\ \pi_*^s \arrow[u, "h"] \end{tikzcd}$$



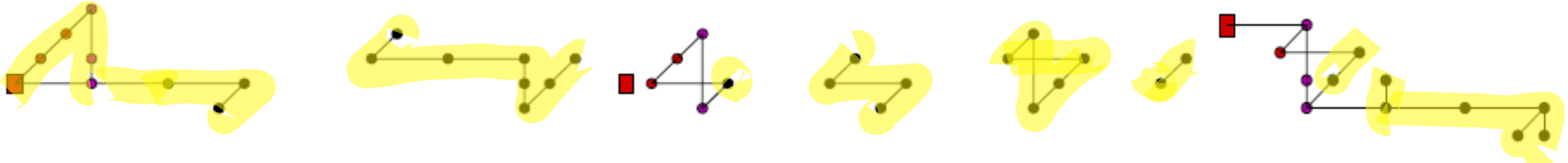
Hurewicz image of TMF (p = 3)

$$\begin{tikzcd}
\pi_* TMF_{(3)} \\
\pi_*^s \arrow[u, "h_{TMF}"']
\end{tikzcd}$$

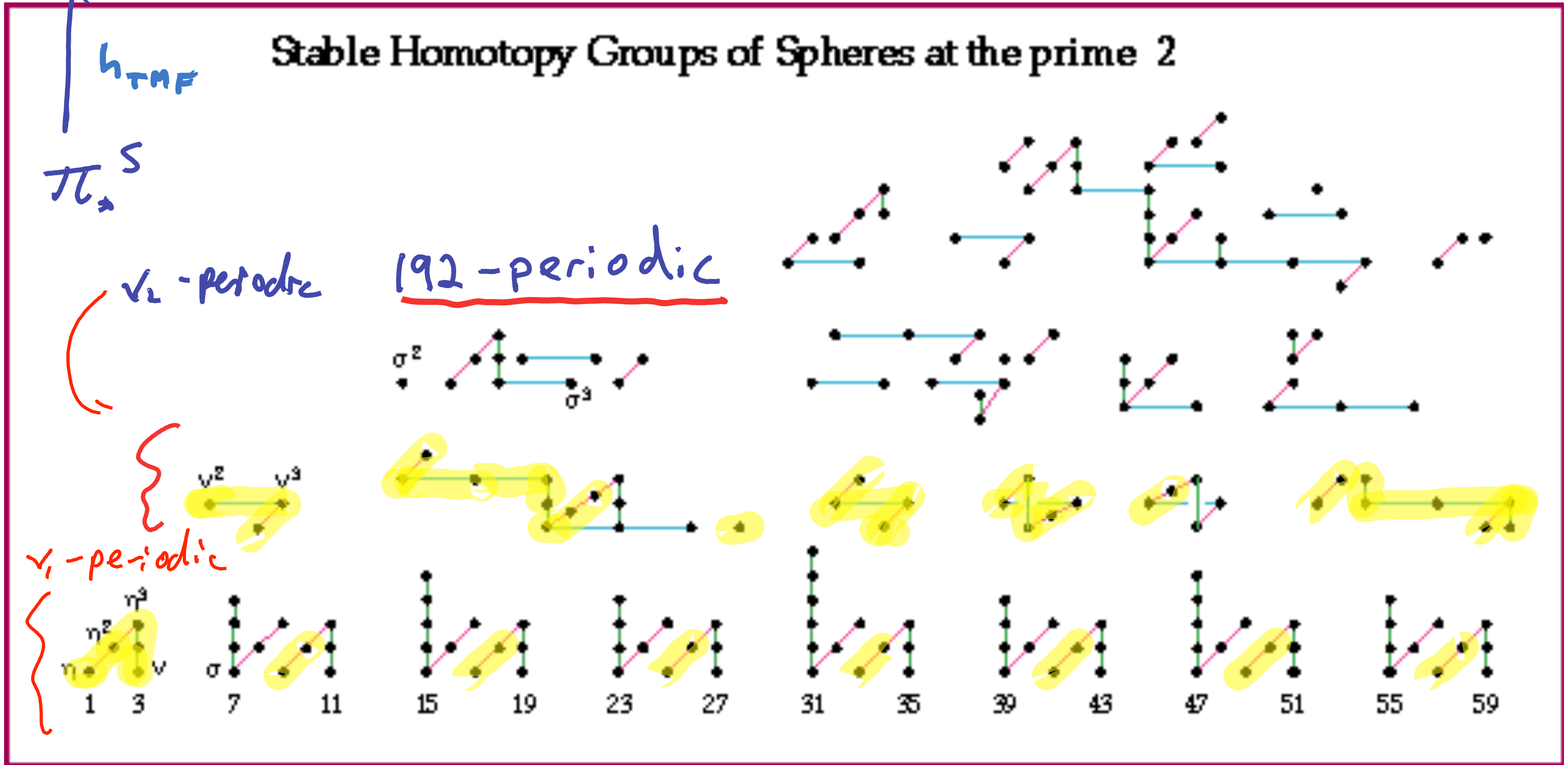
144 periodic



Hurewicz image of TMF (p = 2)



$$\begin{tikzcd}
\pi_*TMF_{(2)} \\
\pi_*^S \arrow[u, "h_{TMF}"]
\end{tikzcd}$$



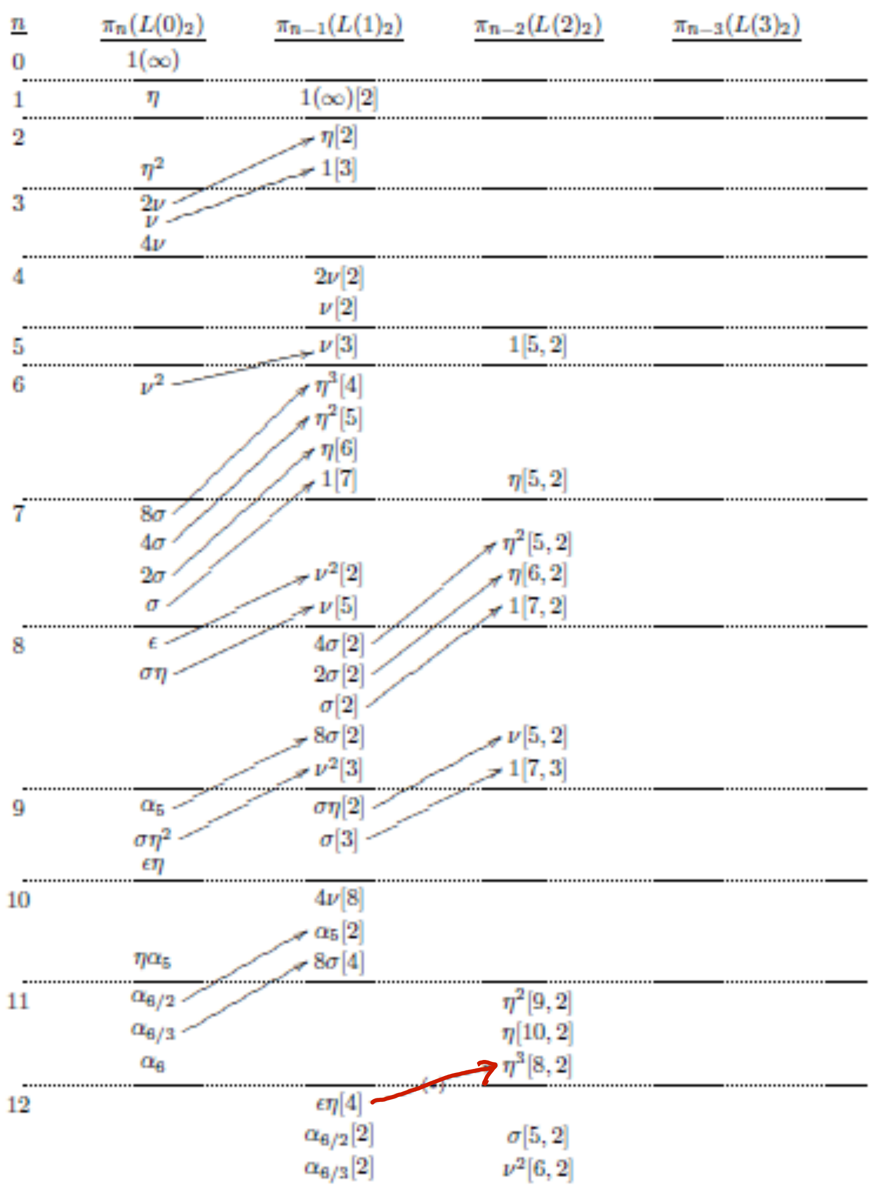
Stable Homotopy Groups of Spheres at the prime 2

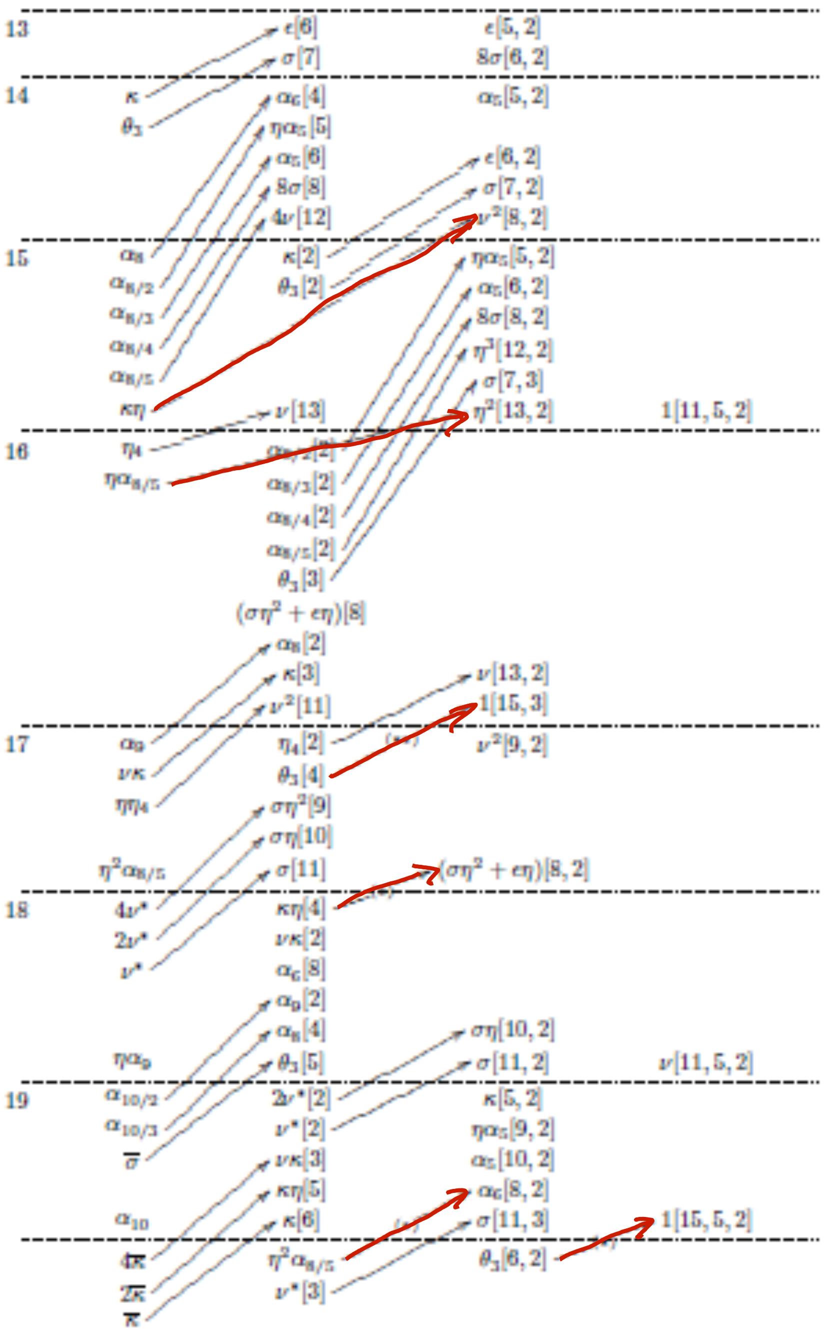
192-periodic

v_2-periodic

v_1-periodic

TABLE 6. The GSS for $\pi_{n+2}(S^2)$





n	$\pi_n(L(0)_2)$	$\pi_{n-1}(L(1)_2)$	$\pi_{n-2}(L(2)_2)$	$\pi_{n-3}(L(3)_2)$
0	$1(\infty)$			
1	η	$1(\infty)[2]$		
2	η^2	$\eta[2]$ $1[3]$		
3	2ν ν 4ν			
4		$2\nu[2]$ $\nu[2]$		
5		$\nu[3]$	$1[5,2]$	
6	ν^2	$\eta^3[4]$ $\eta^2[5]$ $\eta[6]$ $1[7]$	$\eta[5,2]$	
7	8σ 4σ 2σ σ	$\nu^2[2]$ $\nu[5]$	$\eta^2[5,2]$ $\eta[6,2]$ $1[7,2]$	
8	ϵ $\sigma\eta$	$4\sigma[2]$ $2\sigma[2]$ $\sigma[2]$ $8\sigma[2]$ $\nu^2[3]$	$\nu[5,2]$ $1[7,3]$	
9	α_5 $\sigma\eta^2$ $\epsilon\eta$	$\sigma\eta[2]$ $\sigma[3]$		
10	$\eta\alpha_5$	$4\nu[8]$ $\alpha_5[2]$ $8\sigma[4]$		
11	$\alpha_{6/2}$ $\alpha_{6/3}$ α_6		$\eta^2[9,2]$ $\eta[10,2]$ $\eta^3[8,2]$	
12		$\epsilon\eta[4]$ $\alpha_{6/2}[2]$ $\alpha_{6/3}[2]$	$\sigma[5,2]$ $\nu^2[6,2]$	
13		$\epsilon[6]$ $\sigma[7]$	$\epsilon[5,2]$ $8\sigma[6,2]$	
14	κ θ_3	$\alpha_6[4]$ $\eta\alpha_5[5]$ $\alpha_5[6]$ $8\sigma[8]$ $4\nu[12]$	$\alpha_5[5,2]$ $\epsilon[6,2]$ $\sigma[7,2]$ $\nu^2[8,2]$	
15	α_8 $\alpha_{8/2}$ $\alpha_{8/3}$ $\alpha_{8/4}$ $\alpha_{8/5}$ $\kappa\eta$	$\kappa[2]$ $\theta_3[2]$ $\nu[13]$	$\eta\alpha_5[5,2]$ $\alpha_5[6,2]$ $8\sigma[8,2]$ $\eta^3[12,2]$ $\sigma[7,3]$ $\eta^2[13,2]$	$1[11,5,2]$
16	η_4 $\eta\alpha_{8/5}$	$\alpha_{8/2}[2]$ $\alpha_{8/3}[2]$ $\alpha_{8/4}[2]$ $\alpha_{8/5}[2]$ $\theta_3[3]$ $(\sigma\eta^2+\epsilon\eta)[8]$ $\alpha_8[2]$ $\kappa[3]$ $\nu^2[11]$	$\nu[13,2]$ $1[15,3]$	
17	α_9 $\nu\kappa$ $\eta\eta_4$ $\eta^2\alpha_{8/5}$	$\eta_4[2]$ $\theta_3[4]$ $\sigma\eta^2[9]$ $\sigma\eta[10]$ $\sigma[11]$	$\nu^2[9,2]$ $(\sigma\eta^2+\epsilon\eta)[8,2]$	
18	$4\nu^*$ $2\nu^*$ ν^* $\eta\alpha_9$	$\kappa\eta[4]$ $\nu\kappa[2]$ $\alpha_6[8]$ $\alpha_9[2]$ $\alpha_8[4]$ $\theta_3[5]$	$\sigma\eta[10,2]$ $\sigma[11,2]$	$\nu[11,5,2]$
19	$\alpha_{10/2}$ $\alpha_{10/3}$ $\overline{\sigma}$ α_{10}	$2\nu^*[2]$ $\nu^*[2]$ $\nu\kappa[3]$ $\kappa\eta[5]$ $\kappa[6]$	$\kappa[5,2]$ $\eta\alpha_5[9,2]$ $\alpha_5[10,2]$ $\alpha_6[8,2]$ $\sigma[11,3]$	$1[15,5,2]$
	$4\overline{\kappa}$ $2\overline{\kappa}$ $\overline{\kappa}$	$\eta^2\alpha_{8/5}$ $\nu^*[3]$	$\theta_3[6,2]$	

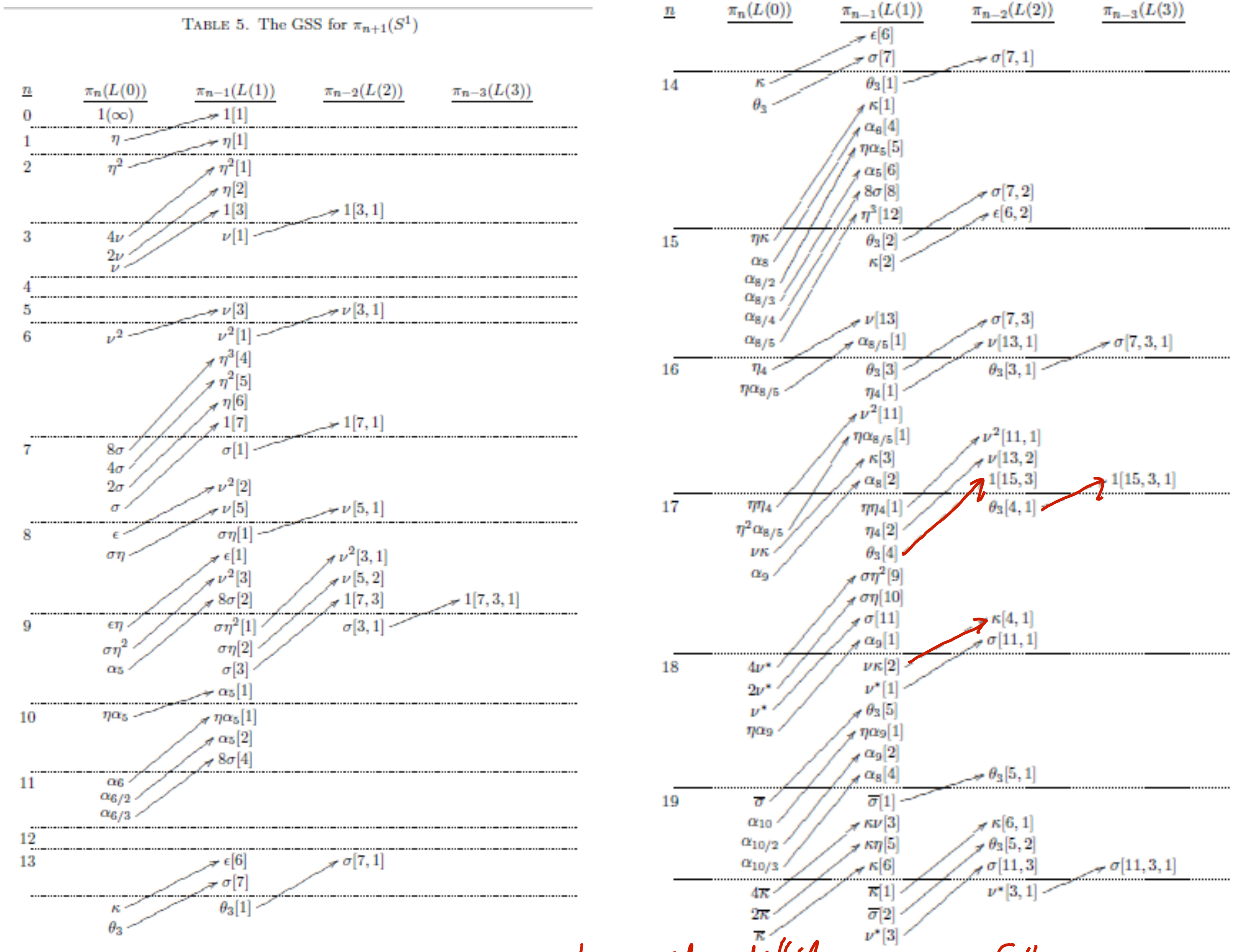
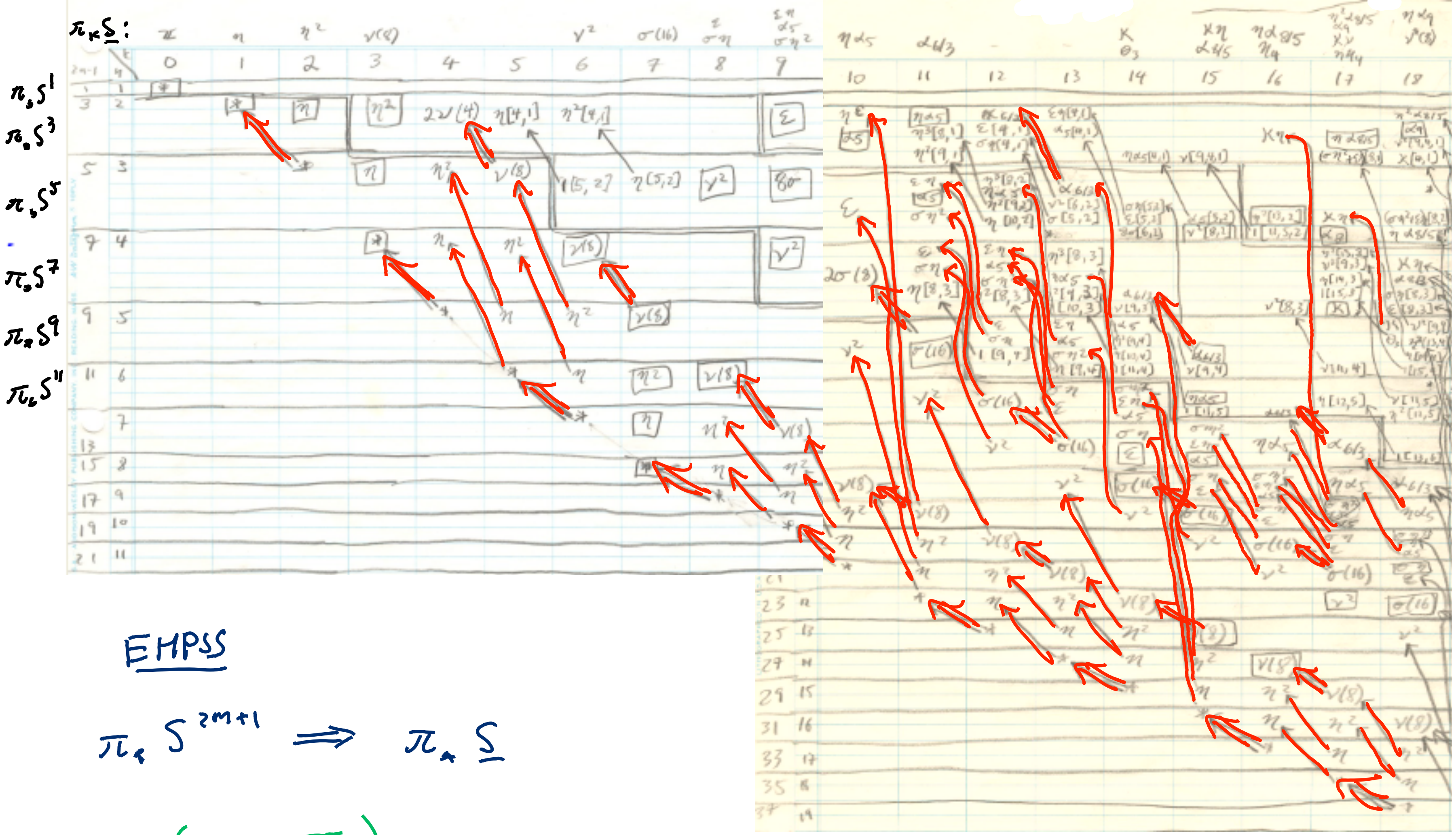


TABLE 5. The GSS for $\pi_{n+1}(S^1)$

$\underline{n}$	$\pi_n(L(0))$	$\pi_{n-1}(L(1))$	$\pi_{n-2}(L(2))$	$\pi_{n-3}(L(3))$
0	$1(\infty)$	$1[1]$		
1	η	$\eta[1]$		
2	η^2	$\eta^2[1]$ $\eta[2]$ $1[3]$	$1[3,1]$	
3	4ν 2ν ν	$\nu[1]$		
4				
5		$\nu[3]$	$\nu[3,1]$	
6	ν^2	$\nu^2[1]$ $\eta^3[4]$ $\eta^2[5]$ $\eta[6]$ $1[7]$	$1[7,1]$	
7	8σ 4σ 2σ σ	$\sigma[1]$ $\nu^2[2]$ $\nu[5]$	$\nu[5,1]$	
8	ϵ $\sigma\eta$	$\sigma\eta[1]$ $\epsilon[1]$ $\nu^2[3]$ $8\sigma[2]$	$\nu^2[3,1]$ $\nu[5,2]$ $1[7,3]$	$1[7,3,1]$
9	$\epsilon\eta$ $\sigma\eta^2$ α_5	$\sigma\eta^2[1]$ $\sigma\eta[2]$ $\sigma[3]$ $\alpha_5[1]$	$\sigma[3,1]$	
10	$\eta\alpha_5$	$\eta\alpha_5[1]$ $\alpha_5[2]$ $8\sigma[4]$		
11	α_6 $\alpha_{6/2}$ $\alpha_{6/3}$			
12				
13		$\epsilon[6]$ $\sigma[7]$	$\sigma[7,1]$	
	κ θ_3	$\theta_3[1]$		

$\underline{n}$	$\pi_n(L(0))$	$\pi_{n-1}(L(1))$	$\pi_{n-2}(L(2))$	$\pi_{n-3}(L(3))$
		$\epsilon[6]$ $\sigma[7]$	$\sigma[7,1]$	
14	κ θ_3	$\theta_3[1]$ $\kappa[1]$ $\alpha_6[4]$ $\eta\alpha_5[5]$ $\alpha_5[6]$ $8\sigma[8]$ $\eta^3[12]$	$\sigma[7,2]$ $\epsilon[6,2]$	
15	$\eta\kappa$ α_8 $\alpha_{8/2}$ $\alpha_{8/3}$ $\alpha_{8/4}$ $\alpha_{8/5}$	$\theta_3[2]$ $\kappa[2]$ $\nu[13]$ $\alpha_{8/5}[1]$	$\sigma[7,3]$ $\nu[13,1]$	$\sigma[7,3,1]$
16	η_4 $\eta\alpha_{8/5}$	$\theta_3[3]$ $\eta_4[1]$ $\nu^2[11]$ $\eta\alpha_{8/5}[1]$ $\kappa[3]$ $\alpha_8[2]$	$\theta_3[3,1]$ $\nu^2[11,1]$ $\nu[13,2]$ $1[15,3]$	$1[15,3,1]$
17	$\eta\eta_4$ $\eta^2\alpha_{8/5}$ $\nu\kappa$ α_9	$\eta\eta_4[1]$ $\eta_4[2]$ $\theta_3[4]$ $\sigma\eta^2[9]$ $\sigma\eta[10]$ $\sigma[11]$ $\alpha_9[1]$	$\theta_3[4,1]$ $\kappa[4,1]$ $\sigma[11,1]$	
18	$4\nu^*$ $2\nu^*$ ν^* $\eta\alpha_9$	$\nu\kappa[2]$ $\nu^*[1]$ $\theta_3[5]$ $\eta\alpha_9[1]$ $\alpha_9[2]$ $\alpha_8[4]$	$\theta_3[5,1]$	
19	$\overline{\sigma}$ α_{10} $\alpha_{10/2}$ $\alpha_{10/3}$	$\overline{\sigma}[1]$ $\kappa\nu[3]$ $\kappa\eta[5]$ $\kappa[6]$	$\kappa[6,1]$ $\theta_3[5,2]$ $\sigma[11,3]$	$\sigma[11,3,1]$
	$4\overline{\kappa}$ $2\overline{\kappa}$ $\overline{\kappa}$	$\overline{\kappa}[1]$ $\overline{\sigma}[2]$ $\nu^*[3]$	$\nu^*[3,1]$	

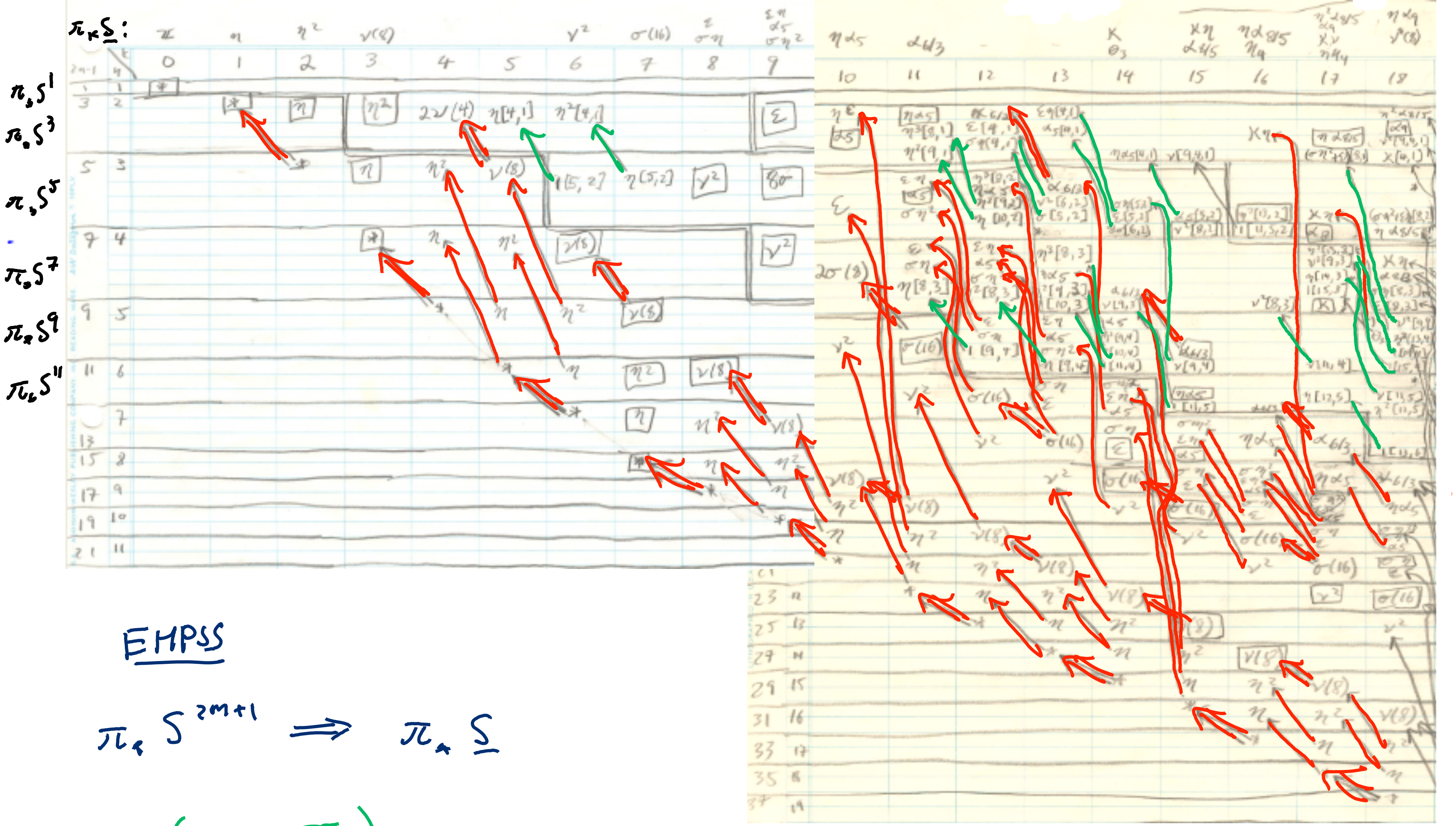
only red diff'ls don't follow



EHPSS

$\pi_* S^{2m+1} \Longrightarrow \pi_* \underline{S}$

$(* =: \mathbb{Z})$



EHPSS

$$\pi_* S^{2m+1} \Rightarrow \pi_* \underline{S}$$

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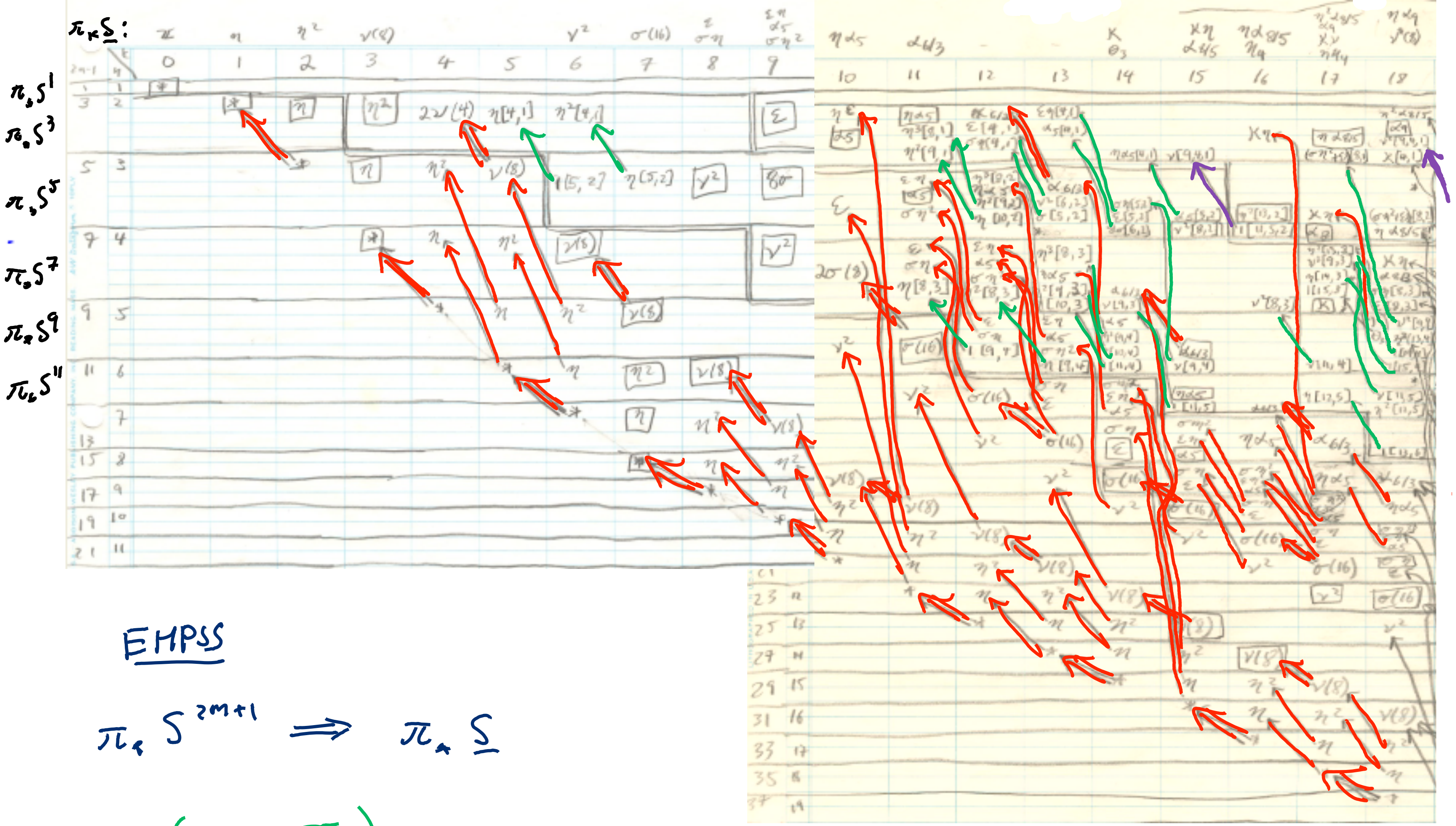

<u>EHPSS</u>

$$\pi_* S^{2m+1} \Rightarrow \pi_* \underline{S}$$

$$(* =: \mathbb{Z})$$

MSRI - Computations in the htpy gps of spheres

Note Title 1/28/2014

$[\pi_*(S^n)]$ $\quad$ $[\pi_n(S^n) = \mathbb{Z}]$ $\quad$ $S^n \xrightarrow[k]{} S^n$

$$\left[\pi_{4n-1}(S^{2n}) \xrightarrow[HI]{} \mathbb{Z}\right]$$

$$S^{4n-1} \xrightarrow[\alpha]{} S^{2n} \longrightarrow C\alpha$$

$H^* C\alpha$ $\qquad$ $\mathbb{Z}\{y\}$ $4n$ $\qquad$ $x^2 = \pm HI(\alpha) y$

$\qquad\qquad\qquad$ $\mathbb{Z}\{x\}$ $2n$

$(\mathbb{R})$ $\quad$ $S^1 \xrightarrow{\cdot 2} S^1$

e.g.

$(\mathbb{C})$

$$\begin{tikzcd} S^3 \arrow[d, phantom, "\cap"] \arrow[drr, "\eta"] & & \\ \mathbb{C}^2 - \{0\} \arrow[r] & \mathbb{C}P^1 \approx S^2 \end{tikzcd}$$

$HI = 1$

$(\mathbb{H})$ $\quad$ $S^7 \xrightarrow[\nu]{} S^4$

$(\mathbb{O})$ $\quad$ $S^{15} \xrightarrow[\sigma]{} S^8$

Thm: (Adams '60) These are only HI 1 elements

[Stable htpy gps]

J-Homo

$$SO(n) \longrightarrow \Omega^n S^n$$

$$(A : \mathbb{R}^n \to \mathbb{R}^n) \longmapsto (A^+ : S^n \to S^n)$$

$$\pi_* SO \xrightarrow{J} \Omega^\infty \Sigma^\infty S^0 = \pi_* QS^0 = \pi_*^s$$

$\mathbb{Z}/2$ 0 $\mathbb{Z}$ 0 0 0 $\mathbb{Z}$ $\mathbb{Z}/2$ $\mathbb{Z}/2$ 0 $\mathbb{Z}$ 0 0 0 $\mathbb{Z}$ --

$\mathbb{Z}/2$'s map in non-trivially

Thm (Adams '66 +) $\# \left| \mathrm{Im}\, J_{4k-1} \right| = \mathrm{denom}\left(\frac{B_{2k}}{4k} \right)$ [J p=2] [J p=3] [J p=5]

EHP Sequence (p=2)

$$\Omega^{n-1} S^{2n-1} \xrightarrow[P]{} \Omega^{n-1} S^{n-1} \xrightarrow[E]{} \Omega^n S^n \xrightarrow[H]{} \Omega^n S^{2n-1}$$

$$H : \pi_{2n-1} S^n \xrightarrow[HI]{} \pi_{2n-1} S^{2n-1} = \mathbb{Z}$$

apply π_* get SS

$$E_1 = \pi_{k+n} S^{2n-1} \Longrightarrow \pi_k^s$$ [EHPSS]

[Curtis alg]

$$HI(\alpha) \Longrightarrow \alpha$$

Diff'ls in EHPSS

$$\begin{tikzcd}
\Omega^n S^n \arrow[r] \arrow[d] & \Omega^{n+1} S^{n+1} \arrow[r] \arrow[d] & \Omega^{n+1} S^{2n+1} \arrow[d] \\
QRP^{n-1} \arrow[r] & QRP^n \arrow[r] & QS^n
\end{tikzcd}$$

$\Longrightarrow$ Map of SS's

$$\begin{tikzcd}
\bigoplus_{n\geq 0} \pi_{n+1+k} S^{2n+1} \arrow[r, Rightarrow, "EHPSS"] \arrow[d] & \pi_k^s \arrow[d, "J\text{-}H"] \\
\bigoplus_{n\geq 0} \pi_k^s(S^n) \arrow[r, Rightarrow, "AHSS"'] & \pi_k^s RP^\infty_+
\end{tikzcd}$$

[Lifting differentials]

Aside Mahowald Invariant

AHSS interpolates

$$\bigoplus_{n\in \mathbb{Z}} \pi_k^s(S^n) \Longrightarrow \pi_k^s RP^\infty_{-\infty} \overset{\text{thm [Lin '80]}}{\simeq} \pi_k^s(S^{-1})$$

$$MI(\alpha) \Longrightarrow \alpha$$

Note! $\deg(HI(\alpha)) < \deg(\alpha)$

$\deg(MI(\alpha)) > \deg(\alpha)$

$MI(2) = \eta$

$MI(\eta) = \nu$

$MI(\nu) = \sigma$

$E =$ "nice" ring spectrum

$(E_*, E_*E) =$ Hopf algebroid

$E_*E \longrightarrow E_*E \otimes_{E_*} E_*E$ coalgebra

$E_*X \longrightarrow E_*E \otimes_{E_*} E_*X$ comodule

E-based ASS

$$\mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*X) \Rightarrow \pi_{t-s} X^{\wedge}_E$$

{ of comodules

Eg. ASS $E = H\mathbb{F}_p$ $E_*E = A_*$ (dual Steenrod alg)

[ASS]

[ASS H.I. 1] $HI(x) = 1$ $\Longleftrightarrow$ x detected by h_i in ASS

[ASS Im J] $d_2(h_i) = h_{i-1}^2 h_0$

$i \geq 4$

[ASS Kervaire]

Kervaire Invariant

$\pi_n^s \cong \Omega_n^{fr}$ $\qquad$ Θ_n = gp of exotic n-spheres

$n \not\equiv 2\ (4)$ $\qquad \Theta_n^{fr} \twoheadrightarrow \pi_n^s / \mathrm{Im}\, J_n$

$n \equiv 2\ (4)$ $\qquad \Theta_{4n-2} \longrightarrow \pi_n^s \xrightarrow[K.I.]{} \mathbb{Z}/2$

Thm (Browder '69)

$x \in \pi_n^s$ has $K.I.(x) = 1$

$\iff$ x detected by h_i^2

in ASS $\qquad (n = 2^i - 2)$

h_i^2 P.C. for $i \leq 5$ $\qquad$ [$i = 5$ Barratt-Jones-Mahowald '85]

Thm [Hill-Hopkins-Ravenel '09]

h_i^2 is NOT a P.C. for $i \geq 7$

Q: is h_6^2 a P.C. ?

Q: what is $d_r(h_i^2)$?

Thm [Mahowald '77]

$\eta_j = h_1 h_j$ is a P.C.

$j \geq 2$

[Q: is $h_0 h_j$ P.C. for $p \geq 3$?]

Thm [Mahowald - Ravenel ?] ('88)

$$HI(h_j; x) = MI(x)$$

e.g. $HI(n_j) = HI(h_j; n) = MI(n)$

$= \nu$

∞'ly many elts w/ $HI = \nu$

$E = MU_{(p)}$ $\Rightarrow$ Adams-Novikov Spectral Sequence

or

BP $\quad BP_* = \mathbb{Z}_{(p)}[v_1, v_2, ---]$ $\quad |v_n| = 2(p^n - 1)$

[p=2 ANSS] $\quad \mathrm{Ext}_{BP_*BP}(BP_*, BP_*) \Rightarrow \pi^S_{*(p)}$

[p=2 Im J ANSS]

[p=2 Kervaire ANSS]

[p=3 ANSS]

(p=3 Im J ANSS)

(p=3 ANSS diffs in low degrees]

[p=3 ASS diffs in low degrees]

Thm [Miller '81]

$p \geq 3$

$$\begin{tikzcd} \mathrm{Ext}_{A_*} \arrow[r, Rightarrow] \arrow[rr, bend left, "ASS"] & \mathrm{Ext}_{BP_*BP} \arrow[r, Rightarrow, "ANSS"'] & \pi \end{tikzcd}$$

[p=3 Toda diff'l]

Thm [Toda '68]

$$d_r(\beta_{p/p}) = \beta_1^p \alpha_1$$

β_{p^i/p^i} = "odd primary KI $\underline{1}$ elts"

Thm [Ravenel '78] $p \geq 5$

β_{p^i/p^i} is Not a P.C. in ANSS, $i \geq 1$

Q: what about $p = 3$?

Chromatic Thy

Motivation: height filtration on $\mathcal{M}_{FG}$ [Morava]

Chromatic SS (Miller-Ravenel-Wilson '77) v_n-periodic $\mathrm{Ext}(M_n)$

$$\bigoplus_n \mathrm{Ext}^{s,t}_{BP_*BP}\left(BP_*, BP_*/(p^\infty, v_1^\infty, \dots, v_{n-1}^\infty)[v_n^{-1}]\right)$$

$$\Longrightarrow \mathrm{Ext}^{s+n,t}_{BP_*BP}(BP_*, BP_*)$$

Greek letter elts $s = 0$

[$p=5$ v_1-periodic]

[$p=5$ v_2-periodic]

[$p=5$ v_3-periodic]

[Greek letter elts]

<u>Geometric Construction</u> $I = (p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$

$BP_* M_I = BP_*/I$

$$\Sigma^{i_n|v_n|} M_I \xrightarrow[v]{} M_I$$ v_n - self map

induces $\cdot v_n^{i_n}$ on BP_*-

$$\begin{tikzcd} S^{si_n|v_n|} \arrow[r] \arrow[rrr, bend right, "\alpha^{(n)}_{si_n/i_{n-1},\ldots,i_0}"'] & \Sigma^{si_n|v_n|} M_I \arrow[r, "v^s"'] & M_I \arrow[r] & S^{top\text{-}cell} \end{tikzcd}$$

<u>Thm</u> [Hopkins-Smith]

I, i_n as above exist asymptotically.

<u>Rmk</u> $HI(\alpha_{i/j}) = \alpha_{i-j}$ "$MI(\alpha_{i/j}) = \beta_{i/j}$"

"$HI(\beta_{i/j,k}) = \beta_{i-j,k}$"

$\mathrm{Ext}(M_1) = \mathrm{Im}\, J$

$\mathrm{Ext}^0(M_2)$ [Miller-Ravenel-Wilson '77]
$p \geq 3$
[Shimomura $p=2$]

[β-family]

$\mathrm{Ext}^*(M_2)$ [Shimomura-Yabe $p \geq 5$ '95, B'12]

[Shimomura-Wang '02, Goerss-Henn-Karamanov-Mahowald
$p=3$ "in progress"]

Q: $p=2$? < Shimomura-Wang '03
{Students of Paul}

Relation to Localizations

$E(n)_* = \mathbb{Z}_{(p)}[v_1, \ldots, v_n, v_n^{-1}]$

$$\begin{tikzcd}
& S_{(p)} \arrow[d] & \simeq & \lim S_{E(n)} \arrow[l] \\
& \vdots \arrow[d] & & \\
M_2S \arrow[r] & S_{E(2)} \arrow[d] & & \\
M_1S \arrow[r] & S_{E(1)} \arrow[d] & & \\
& S_{\mathbb{Q}} & &
\end{tikzcd}$$

chromatic Convergence Thm
[Hopkins-Ravenel '92]

ANSS

$\mathrm{Ext}(M_n) \Rightarrow \pi_* M_nS$

$$v_n^{-1} M_I \longrightarrow (M_I)_{E(n)}$$

Telescope conj

this map is an equiv.

True for $n=1$ [Miller '81 $p \geq 3$
Mahowald '81 $p=2$]

Consequence

$$\varinjlim_{I} v_n^{-1} M_I \simeq M_n S$$

Translating

(1) Suppose $\{v_n^{s d_n} x\} \subset \pi_* S_{E(n)}$

is a v_n-periodic family

Then $v_n^{s d_n} x$ comes from

$\pi_* S$ for $s \gg 0$.

(2) All v_n-periodicity in

π_*^S is detected in

$\pi_* S_{E(n)}$

[Believed to be false for $n \geq 2$]