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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Leanne Merrill Email/Phone: learnerm@uoregon.edu/ 5184617614

Speaker's Name: Daniel Isaksen

Talk Title: Computations in Motivic Homotopy Theory

Date: 1/29/14 Time: 10:30 am / pm (circle one)

List 6-12 key words for the talk: stable motivic homotopy theory, realization functors, 2-complete motivic Adams spectral sequence, classical homotopy groups, eta-local sphere.

Please summarize the lecture in 5 or fewer sentances: This lecture talked generally about motivic homotopy theory and performed some calculations on homotopy groups of spheres in this context. This allowed conclusions about classical homotopy groups of spheres, surprisingly contradicting earlier calculations. It finished by talking about the eta-local sphere.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- [x] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [x] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- [x] For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- [x] When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- [x] Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

This talk is supplemented by several slides which follow after these notes.

(I) Motivic homotopy theory.

Start with Sm/k = smooth varities over k.

Not enough gluing constructions in this category.

(1) Formally adjoin homotopy colimits to get simplicial presheaves.

(2) Restore some desired relations:

If $\{U, V\}$ is a cover of X, then declare that

$$\begin{array}{ccc} U \cap V & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

is a homotopy pushout.

(This gives us things like Mayer-Vietoris).

(3) Declare that these projections are equivalences:

$$X \times \mathbb{A}^1 \longrightarrow X$$

This gives unstable motivic homotopy theory.

Motivic homotopy theory is, in principal, about simplicial presheaves, but in practice it is not. Easy to get bogged down in the machinery.

Ex $\dfrac{\mathbb{A}^1}{\mathbb{A}^1 \setminus 0} \simeq \Sigma(\mathbb{A}^1 \setminus 0)$ (did not need to delve into details).

Stable motivic homotopy theory:

$S^{1,0}$ = simplicial circle

$S^{1,1}$ = $\mathbb{A}^1 \setminus 0$.

want to stabilize with respect to both.

This gives stable motivic homotopy theory.

Warning: Not every spectrum is built out of spheres.

Focus on "cellular" motivic homotopy theory.

(Downside: will lose some examples (ie elliptic curves))

Theory of "motives":

A motive is supposed to retain just the cohomological information of a variety X.

$H\mathbb{Z} \wedge \Sigma^\infty X$ is exactly the object, so it is the motive of X.

Question: why not just study $\Sigma^\infty X$?

Realization functors:

(motivic homotopy / $\mathbb{C}$) → (classical homotopy)

(motivic homotopy / $\mathbb{R}$) → ($\mathbb{Z}/2$-equivariant homotopy)

$S^{1,0} \longmapsto S^{1,0}$ (trivial action)

$S^{1,1} \longmapsto S^{1,1}$ (non-trivial reflection action)

If you know something about one side, you may be able to deduce something about the other.

(II) Survey of computations of $\pi_{**}(S^{0,0})$.

- $\pi_{p,q} = 0$ if $q > p$ (Morel).
- $\pi_{p,p}$ in terms of "Milnor - Witt K-theory"
  (contains number theoretic info about the underlying field.)

  This is also due to Morel.

- $\pi_{p+1,p}$ for "low dimensional" fields (Ormsby - Østvær).

  This gets less complete after this stage.

- Have $\eta \in \pi_{1,1}$, $\nu \in \pi_{3,2}$, $\sigma \in \pi_{7,4}$ and
  $\eta\nu = 0$, $\nu\sigma = 0$ (Dugger - I.)

- $\pi_{p,0}$ over $\mathbb{C}$ is the same as classical π_p (Levine)

- 2-complete motivic ANSS over $\mathbb{C}$
  (Hu - Kriz - Ormsby)

Focus on these for the rest of the talk:

- 2-complete motivic Adams SS over $\mathbb{C}$
  (Dugger, Guillou, I.)

- 2-complete motivic Adams SS over $\mathbb{R}$
  (Dugger, Hill, I.)

(III) Motivic Adams Spectral Sequence (2-complete)

Algebraic input: (Voevodsky)

$$H^{**}(pt) = \mathbb{M}_2 = \mathbb{F}_2[\tau], \quad |\tau| = (0,1).$$

$$\text{(dual) Steenrod algebra} = \mathbb{M}_2[\tau_i, \xi_i] \big/ \tau_i^2 = \tau \xi_{i+1}$$

<u>Note</u>: Inverting τ corresponds to recovering classical version.

Then:

$$E_2 = \mathrm{Ext}_A(\mathbb{M}_2, \mathbb{M}_2).$$

(see chart) page 3/4. This calculation is entirely tractable by computer.

Notes about picture:
- each black dot is a copy of $\mathbb{M}_2$.
- red dots are $\mathbb{M}_2/\tau$.
- red arrows are infinite towers of h_1 multiplication.
- purple line: $h_0^2 h_2 = \tau h_1^3$ so $0 = \tau h_1^4$.
- purple lines are multiples of τs.

Next step is to calculate Adams differentials (see chart, page 4/4).

But these computations gave some information about classical homotopy groups.

(IV) Classical Homotopy Groups:

(see chart, page 214).

Notes on picture: Hatcher's charts are wrong in 56 stem, 62 stem, others! — indicated by green dashes, blue differentials.

Red circles are multiples of h_4^3 class.

Everything above purple lines is detected by TMF.

(V) η-local sphere.

$$S^{0,0}[\eta^{-1}] = \operatorname{colim}(S \xrightarrow{\eta} S \xrightarrow{\eta} S \xrightarrow{\eta} \cdots)$$

Adams spectral sequence:

$$\mathrm{Ext}[h_1^{-1}] \Rightarrow \pi_{**} S^{00}[\eta^{-1}]$$

<u>Thm</u> (Guillou – I):

$$\mathrm{Ext}[h_1^{-1}] = \mathbb{F}_2[v_1^4, v_2, v_3, v_4, \ldots]$$

Adams $d_2(v_3) = v_2^2$

$d_2(v_4) = v_3^2$

<u>Conjecture</u>: $d_2(v_n) = v_{n-1}^2$

$$\pi_{**}(S^{0,0}[\eta^{-1}]) = \mathbb{F}_2[\eta^{\pm}, \beta, \varepsilon]/\varepsilon^2$$

β in (8,4), ε in (8,5)

(Known, to 70 stem).

Other questions: what about nonnilpotent elements?
are there others besides those on
the diagonal computed by Morel?

Changing directions:

Rigid connection between:
motivic Adams over $\mathbb{C}$ and
classical Adams-Novikov.

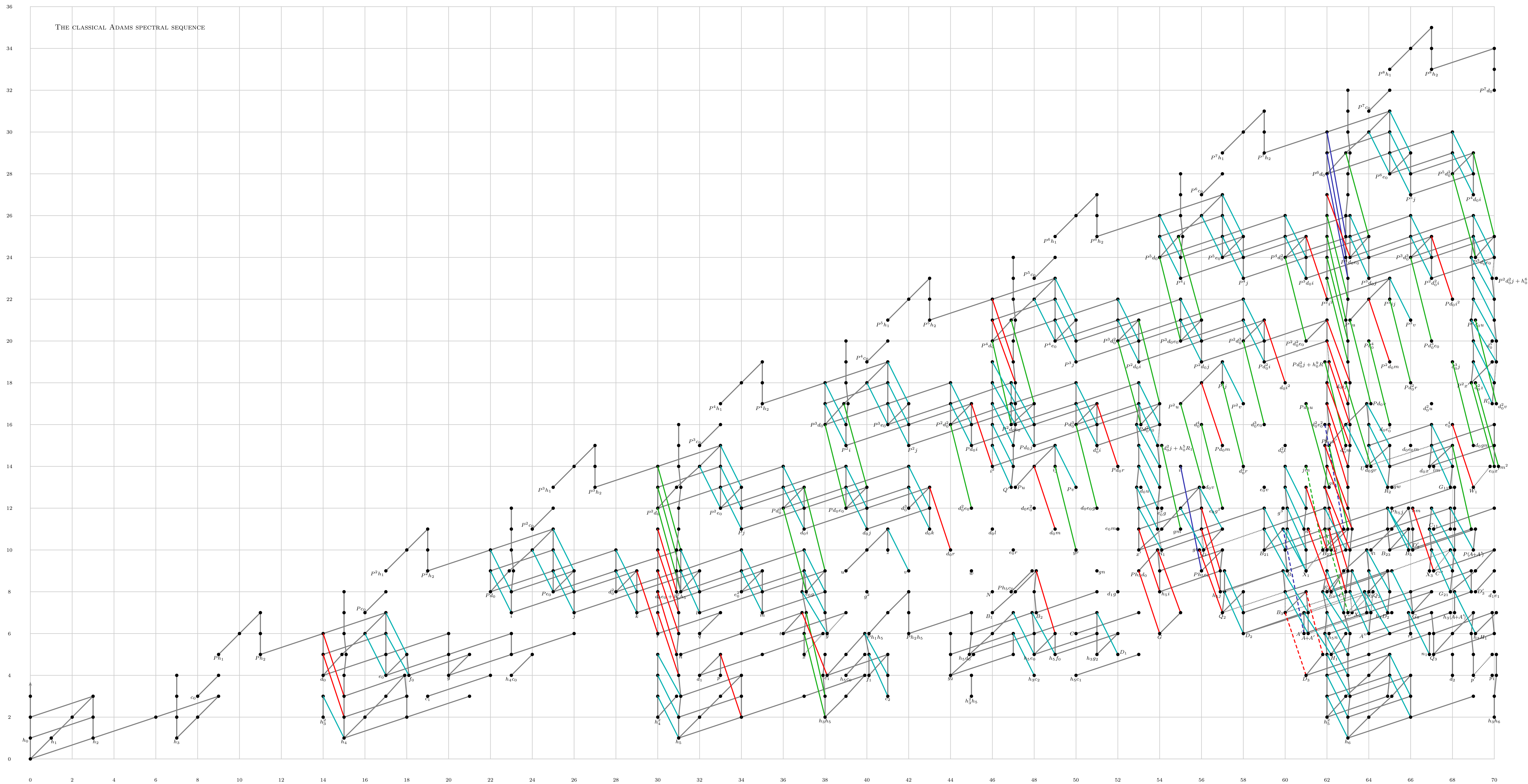
<u>Challenge</u>: Machine compute E_2-ANSS.
(This would make many things easier)

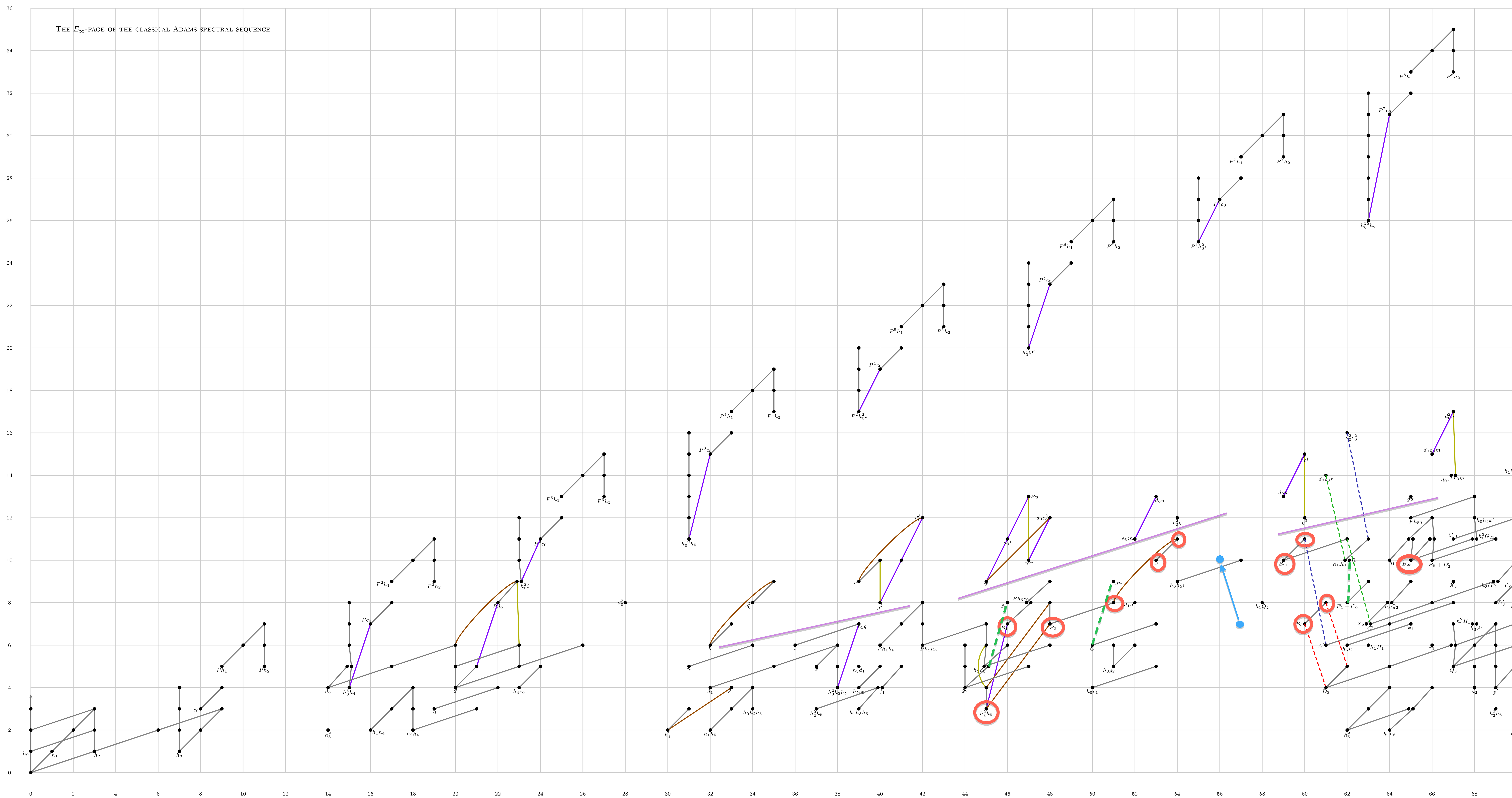
<u>Reward</u>: Get new stable stems.

VI Preliminary calculations over $\mathbb{R}$.

A. The image of J is not what you think.
In particular, $h_0^4 h_3 \neq 0$.
$h_0^8 h_4 \neq 0$.
etc.

Guess: Extra 2 in
dimensions 7 mod 8.


The classical Adams spectral sequence


The E_∞-page of the classical Adams spectral sequence


The cohomology of the motivic Steenrod algebra
0
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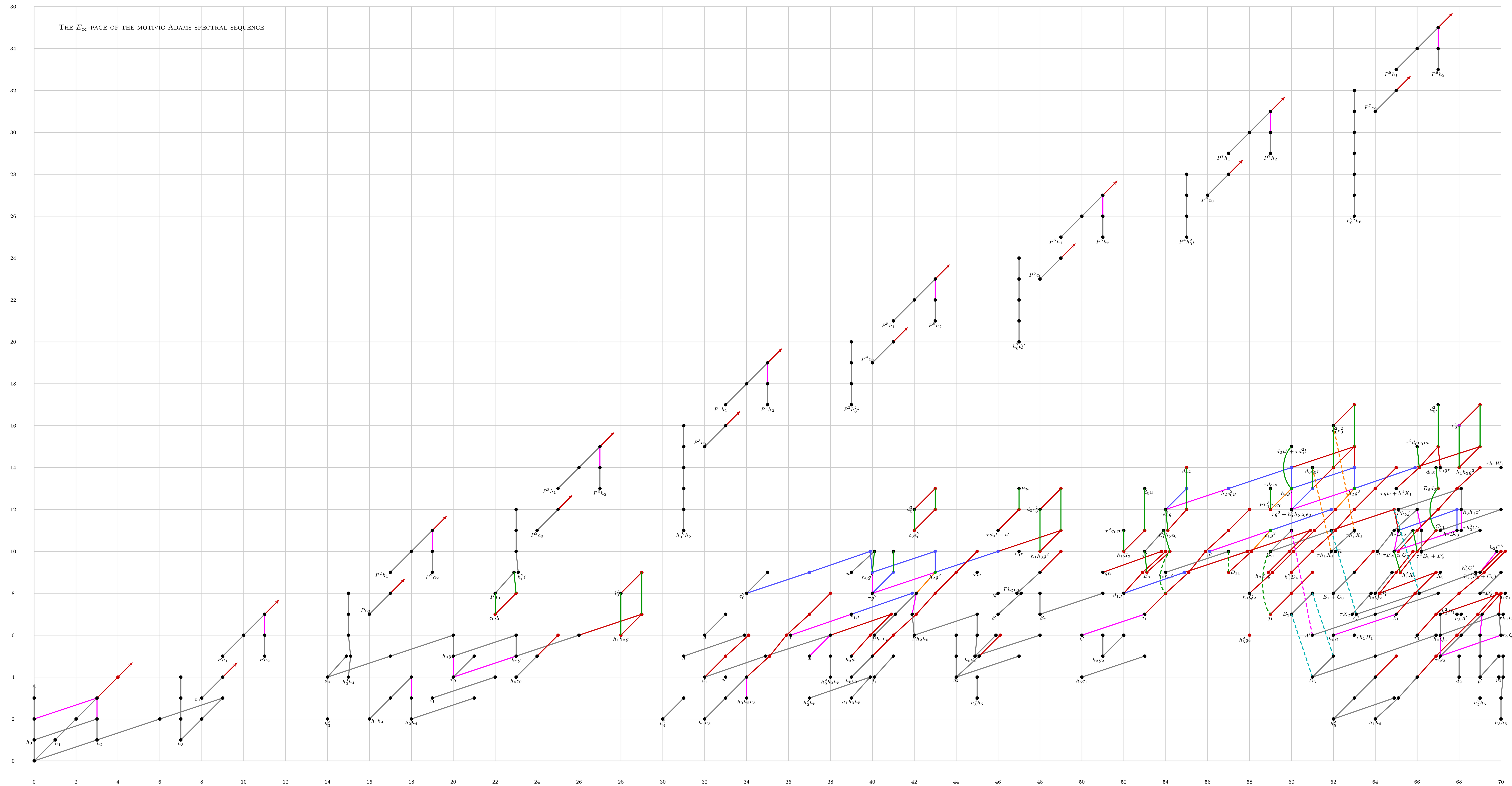
The E_∞-page of the motivic Adams spectral sequence
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40 42 44 46 48 50 52 54 56 58 60 62 64 66 68 70