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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Leanne Merrill Email/Phone: leannem@uoregon.edu / 518 461 7614

Speaker's Name: Craig Westerland

Talk Title: Views on the J-homomorphism

Date: 1 / 29 / 14 Time: 11 : 30 (am) / pm (circle one)

List 6-12 key words for the talk: K-theory, Picard Spectrum, Adams conjecture, finite fields, Morava K-theory, unit spectrum.

Please summarize the lecture in 5 or fewer sentances: This talk introduced some of the many places in which the J-homomorphism from K-theory appears. It started with the classical applications to sphere spectra, and then moved on to unit spectra, picard spectra, and some of the applications to algebraic K-theory of finite fields.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
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Craig Westerland - Views on the J-homomorphism 1/28/14

This is a Beamer Presentation, and these notes will supplement it. The presentation follows.

Supplement to slide 4:

$$\begin{array}{ccc} GL_1 R & \longrightarrow & \Omega^\infty R \\ \downarrow & & \downarrow \pi_0 \\ (\pi_0 R)^\times & \hookrightarrow & \pi_0 R \end{array}$$

describes how $GL_1 R$ becomes a spectrum

Supplement to slide 5:

$X \in$ Groupoid.

So B Groupoid $\ni X$ (classifying space)

Ω_X B Groupoid $\simeq$ Aut (X).

Supplement to slide 7:

Analogy to understand Adams operations:

$(\mathbb{N}, \times) \circlearrowright KO(X)$.

Supplement to slide 8:

$F = Q_{\pm 1} S^0$.

$\cup\!\!|$

$SF = Q_{+1} S^0 \simeq Q_0 S^0$.

Supplement to slide 12:

For $n=1$, $E_1 = \hat{K_p}$

$\mathbb{G}_1 = \mathbb{Z}_p^\times$ (Adams operations)

$K_1 = KU/p$.

so K_n is a generalization of this concept.

Supplement to slide 13:

ρ_n generates a copy of $\mathbb{Z}_p[p^{\pm\frac{1}{n}}]$

inside of $[S\langle \det \pm\rangle^{\otimes j}, R_n]$.

Views on the J-homomorphism

Craig Westerland

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29 January 2014

K-theory and spherical fibrations

Recall: $KO(X) := \{\mathbb{R}\text{-vector bundles over } X, \oplus\}^{gp}$. Similarly,

Definition

$Sph(X) := \{\text{sectioned spherical fibrations over } X, \wedge_X\}^{gp}$. These are fibrations

$$\begin{array}{ccc} S^n & \longrightarrow & E \\ & & p \downarrow \uparrow \sigma \\ & & X \end{array} \qquad n \in \mathbb{N}.$$

Examples:

1. Trivial: $S^n \times X$.
2. Hopf: $\eta : S^3 \to S^2$ is *not* sectioned.
3. Unit sphere bundles: If $W \to X$ is a vector bundle, then $S(W) \to X$ is a spherical fibration. If W admits a nowhere-vanishing section (i.e., $W \cong V \oplus \mathbb{R}$), then $S(W)$ is sectioned.

Note: if V is a vector bundle, then $S(V \oplus \mathbb{R}) \to X$ is the fibrewise 1-point compactification of V.

J-homomorphism and representing spaces

Definition

The *J-homomorphism*

$$J : KO(X) \to Sph(X)$$

sends V to $S(V \oplus \mathbb{R})$.

Recall: KO is represented by $BO \times \mathbb{Z}$, since $O = \lim_n O(n)$ is the (stable) structure group for $\mathbb{R}$-vector bundles. That is, if X is compact,

$$KO(X) \cong [X, BO \times \mathbb{Z}].$$

Similarly: Sph is represented by $BF \times \mathbb{Z}$ for $F = \lim_n F(n)$, where

$$F(n) = \mathrm{hAut}_*(S^n) = \{f : S^n \to S^n, f \text{ is a based homotopy equivalence}\}$$

This is an associative monoid under composition of functions; $\pi_0 F(n) \cong \{\pm 1\}$.

Then J is represented by $J : BO \times \mathbb{Z} \to BF \times \mathbb{Z}$, induced by $O(n) \to F(n)$:

$$(M : \mathbb{R}^n \to \mathbb{R}^n) \mapsto (M \cup \{\infty\} : S^n \to S^n).$$

Unit spectra

Definition

If R is an E_∞ ring spectrum, define the unit space $\mathrm{GL}_1 R$ as the union of components of $\Omega^\infty R$ associated to $(\pi_0 R)^\times \subseteq \pi_0 R$.

$\mathrm{GL}_1 R$ is an E_∞-space with multiplication coming from the product on R. The unit spectrum $\mathrm{gl}_1 R$ is the connective spectrum with $\Omega^\infty \mathrm{gl}_1 R = \mathrm{GL}_1 R$.

Note: For $n \geq 1$, $\pi_n \mathrm{gl}_1 R = \pi_n \mathrm{GL}_1 R \cong \pi_n \Omega^\infty R = \pi_n R$.

Example

$R = S^0$. Then $\pi_0 S^0 = \mathbb{Z} \supseteq \{\pm 1\} = (\pi_0 S^0)^\times$. The zeroth space of the spectrum is $QS^0 = \lim_n \Omega^n S^n$.

$$\mathrm{GL}_1 S^0 = Q_{\pm 1} S^0 = \lim_{n\to\infty} \Omega^n_{\pm 1} S^n = \lim_{n\to\infty} F(n) = F.$$

The products on $\mathrm{GL}_1 S^0$ and F are *not* the same (smash product vs. composition), but do commute, so $B\,\mathrm{GL}_1 S^0 \simeq BF$:

$$Sph_{>0}(X) = [X, BF] = [X, B\,\mathrm{GL}_1 S^0] = [\Sigma^\infty X, \Sigma\,\mathrm{gl}_1 S^0].$$

Picard spectra

Let R be an E_∞-ring spectrum, and $(\mathrm{Mod}_R, \wedge_R)$ be the associated symmetric monoidal ∞-category of its (right) module spectra.

Definition (Ando-Blumberg-Gepner)

The Picard space $\mathrm{Pic}(R) \subseteq \mathrm{Mod}_R$ is the full subgroupoid spanned by the modules M which invertible with respect to $\wedge_R$. This is a grouplike E_∞ space; the Picard spectrum $\mathrm{pic}(R)$ is the associated connective spectrum.

Note: R is the unit of $\otimes_R$, so take $R \in \mathrm{Pic}(R)$ as a basepoint. Then

$$\Omega\,\mathrm{Pic}(R) = \mathrm{Aut}_R(R) = \mathrm{GL}_1(R)$$

In fact, this gives a connected cover $\Sigma\,\mathrm{gl}_1(R) \to \mathrm{pic}(R)$.

The J-homomorphism: in this language is

$$\begin{array}{ccc} bo & \xrightarrow{J} & \Sigma\,\mathrm{gl}_1(S^0) \\ \downarrow & & \downarrow \\ ko & \xrightarrow{J} & \mathrm{pic}(S^0) \end{array}$$

(**Here:** $bo = ko_{>0}$ is the connected cover)

induced by $\mathbb{R}$-Vect $\to \mathrm{Pic}(S^0)$, where $V \mapsto V \cup \{\infty\}$.

Remark: Dustin Clausen has formulated an analogous $(K\mathbb{Q}_p)_{>1} \to \mathrm{pic}(S^0)$.

Image in homotopy

Theorem (Bott periodicity)

k mod 8	1	2	3	4	5	6	7	8
$\pi_k ko$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

The induced map $\pi_* J : \pi_* ko \to \pi_* \operatorname{pic}(S^0) \cong \pi_{*-1}(S^0)$ for $* > 0$ is known:

Theorem (Adams, Quillen)

$\pi_* J$ is an injection if $* = 1, 2 \bmod 8$. Further, in dimension $* = 4n$, $\operatorname{im}(\pi_* J)$ is $\mathbb{Z}/m$ where m is the denominator of $B_{2n}/4n$.

Here, the Bernoulli numbers satisfy

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}$$

Summary of p-torsion: ($p > 2$): If $* = 2(p-1)p^k m$, where m is coprime to p,

$$_p\operatorname{im}(\pi_* J) = \mathbb{Z}/p^{k+1}$$

Otherwise, $_p\operatorname{im}(\pi_* J) = 0$. If $* = 2^k m$ with m odd, then $_2\operatorname{im}(\pi_* J) = \mathbb{Z}/2^{k+1}$.

Adams conjecture

For $k \in \mathbb{N}$, the k^{th} *Adams operation* is a natural transformation

$$\psi^k : KO(X) \to KO(X).$$

Properties:

1. For line bundles L, $\psi^k(L) = L^{\otimes k}$.
2. Each ψ^k is a ring homomorphism.
3. $\psi^k \circ \psi^\ell = \psi^{k\ell}$.

These are represented by maps $\psi^k : BO \to BO$.

Theorem (Quillen, Sullivan, Friedlander)

For a finite CW complex X and $V \in KO(X)$, there exists $e = e(k, V)$ so that $k^e J(V) = k^e J(\psi^k(V)) \in Sph(X)$.

Equivalently, on finite skeleta, the composite map

$$BO \xrightarrow{\psi^k - 1} BO \xrightarrow{J} BF \xrightarrow{loc} BF[\tfrac{1}{k}]$$

is null-homotopic. There exists a complex analogue (for BU), too.

Image of J space/spectra

Definition

For $p = 2$: Let $J_{(2)}$ be the homotopy fibre of the map

$$\psi^3 - 1 : BO_{(2)} \to BSpin_{(2)}.$$

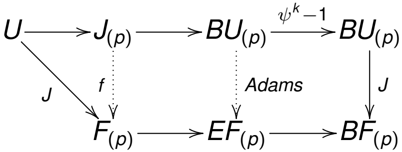
For $p > 2$: choose $k \in \mathbb{N}$ so that k mod p^2 is a generator of $(\mathbb{Z}/p^2)^\times$, and define $J_{(p)}$ to be the homotopy fibre of the map

$$\psi^k - 1 : BU_{(p)} \to BU_{(p)}$$

Write $j_{(2)}$ (respectively $j_{(p)}$) for the associated (ring) spectra. The unit of ko or ku lifts to $e : S^0 \to j_{(p)}$. This gives

$$e : SF \simeq Q_0 S^0 \to J_{(p)}.$$

The Adams conjecture gives us a commuting diagram of fibre sequences:



Note: $k \in \mathbb{Z}_{(p)}^\times$.

Computing the image of J in homotopy

Theorem (Mahowald; May-Tornehave)

The maps e and f split $J_{(p)}$ off of $Q_0 S^0_{(p)}$.

So: the p-torsion in $\mathrm{im}(\pi_* J : \pi_* ko \to \pi_{*-1} S^0)$ is isomorphic to $\pi_{*-1} J_{(p)}$:

$$\cdots \longrightarrow \pi_* J_{(p)} \longrightarrow \pi_* BU_{(p)} \xrightarrow{\psi^k - 1} \pi_* BU_{(p)} \longrightarrow \pi_{*-1} J_{(p)} \longrightarrow \cdots$$

Now, $\pi_* BU = \mathbb{Z}[\beta]$, where $\beta \in \pi_2 BU$ is the Bott periodicity class. Compute: $\psi^k(\beta) = k\beta$, so if $* = 2n$, this is

$$\cdots \longrightarrow \pi_{2n} J_{(p)} \longrightarrow \mathbb{Z}_{(p)} \xrightarrow{k^n - 1} \mathbb{Z}_{(p)} \longrightarrow \pi_{2n-1} J_{(p)} \longrightarrow \cdots$$

So for $n > 0$, $\pi_{2n} J_{(p)} = 0$, and

$$\pi_{2n-1} J_{(p)} = \mathbb{Z}_{(p)}/(k^n - 1) = \begin{cases} 0, & n \neq (p-1)p^s m \\ \mathbb{Z}/p^{s+1}, & n = (p-1)p^s m \end{cases}$$

Recall that k mod p^2 generates $(\mathbb{Z}/p^2)^\times$. Then:

- $k^n - 1$ is a unit in $\mathbb{Z}_{(p)}$ when $k^n \neq 1 \bmod p \iff (p-1) \nmid n$.
- Further, $k^{(p-1)} \in 1 + p\mathbb{Z}_{(p)}$, so $k^{(p-1)p^s m} \in 1 + p^{s+1}\mathbb{Z}_{(p)}$.

Algebraic K-theory of finite fields

Let $q = p^m$, and define $F\psi^q$ to be the homotopy fibre of $\psi^q - 1 : BU \to BU$.

Quillen used Brauer theory to lift the defining representation of $\mathrm{GL}_n(\mathbb{F}_q)$ on $\mathbb{F}_q^n$ to a virtual complex representation, yielding a map

$$B\,\mathrm{GL}_n(\mathbb{F}_q) \to BU$$

Action of ψ^q on $B\,\mathrm{GL}_n(\mathbb{F}_q)$ is the q-Frobenius so this lifts to $F\psi^q$. In the limit:

Theorem (Quillen)

The map $\Omega^\infty K(\mathbb{F}_q) = B\,\mathrm{GL}_\infty(\mathbb{F}_q)^+ \to F\psi^q$ is an equivalence. Hence

$$K_n(\mathbb{F}_q) = \begin{cases} 0, & n = 2i \\ \mathbb{Z}/(q^i - 1), & n = 2i - 1 \end{cases}$$

Interpretation: Let ℓ be prime, and pick $q = p^m$ so that $q \bmod \ell^2$ is a generator of $(\mathbb{Z}/\ell^2)^\times$. Then from Suslin's theorem:

$$\begin{array}{ccccc} j_\ell^\wedge & \longrightarrow & ku_\ell^\wedge & \xrightarrow{\psi^q - 1} & ku_\ell^\wedge \\ \uparrow \simeq & & \uparrow \simeq & & \uparrow \simeq \\ K(\mathbb{F}_q)_\ell^\wedge & \longrightarrow & K(\overline{\mathbb{F}}_q)_\ell^\wedge & \xrightarrow{\psi^q - 1} & K(\overline{\mathbb{F}}_q)_\ell^\wedge \end{array}$$

Note: This exhibits $K(\mathbb{F}_q)_\ell^\wedge$ as the homotopy fixed points $(K(\overline{\mathbb{F}}_q)_\ell^\wedge)^{h\,\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)}$.

$K(1)$-local homotopy

Let

$$K_1 := KU/p = \vee_{i=0}^{p-2} \Sigma^{2i} K(1),$$

Pick $k \in \mathbb{Z}$ which generates $(\mathbb{Z}/p^2)^\times$, and define $\mathbb{J}_p$ by the fibre sequence

$$\mathbb{J}_p \longrightarrow KU_p^\wedge \xrightarrow{\psi^k - 1} KU_p^\wedge$$

Theorem

The unit map $e : S^0 \to \mathbb{J}_p$ is an isomorphism in $K(1)_*$, so $\mathbb{J}_p \simeq L_{K(1)} S^0$.

Here $L_{K(1)} S^0$ is the Bousfield localization of S^0 at $K(1)$.

Idea: Compute $K(1)_* KU_p = C(\mathbb{Z}_p^\times, \mathbb{F}_p)$, and the action of ψ^k is by translation by $k \in \mathbb{Z}_p^\times$. Since $\langle k \rangle \leq \mathbb{Z}_p^\times$ is dense, fixed functions are constants = $\mathrm{im}(e_*)$.

Conclusion: the localization map $S^0 \to L_{K(1)} S^0$ carries

$$\mathrm{im}(\pi_* J) \cong \pi_*(\mathbb{J}_p), \quad * > 0$$

isomorphically onto $\pi_* L_{K(1)} S^0$ in positive degrees.

Note: This presents $L_{K(1)} S^0$ as the homotopy fixed point spectrum $(KU_p^\wedge)^{h\mathbb{Z}_p^\times}$ for an action of $\mathbb{Z}_p^\times$ by a p-adic extension of the Adams operations.

Morava K and E-theories

Definition

- Let E_n denote the Morava E-theory associated to the Lubin-Tate deformation space of the formal group Γ_n over $\mathbb{F}_{p^n}$ with $[p](x) = x^{p^n}$.
- The Morava stabilizer group is $\mathbb{G}_n = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \ltimes \mathrm{Aut}(\Gamma_n)$.
- The Morava K-theories are $K_n = E_n/\mathfrak{m}$, and $K(n) = K_n^{h\,\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)}$.

Theorem (Morava, Goerss-Hopkins-Miller, Devinatz-Hopkins, Behrens-Davis)

$\mathbb{G}_n$ acts on E_n in such a way that $E_n^{h\mathbb{G}_n} \simeq L_{K(n)}S^0$.

There exists a reduced norm $\det_\pm : \mathbb{G}_n \to \mathbb{Z}_p^\times$ coming from the determinant of the action of $\mathbb{G}_n$ on $\mathrm{End}(\Gamma_n)$. Define

- $S\mathbb{G}_n^\pm := \ker(\det_\pm)$, and
- $R_n := E_n^{hS\mathbb{G}_n^\pm}$: determinantal K-theory, half the sphere, or the Iwasawa extension of $L_{K(n)}S^0$.

Then, for a topological generator $k \in \mathbb{Z}_p^\times$, there is a fibre sequence

$$L_{K(n)}S^0 = (E_n^{hS\mathbb{G}_n^\pm})^{h\mathbb{Z}_p^\times} \longrightarrow R_n \xrightarrow{\psi^k - 1} R_n$$

Higher chromatic analogues

Define $S\langle\det_\pm\rangle = \mathrm{hofib}(\psi^k - k)$. Then $S\langle\det_\pm\rangle \in \mathrm{Pic}_n = \mathrm{Pic}(L_{K(n)}\mathrm{Spectra})$, and

$$(E_n)_* S\langle\det_\pm\rangle \cong (E_n)_*[\det_\pm].$$

When $n = 1$, $S\langle\det_\pm\rangle = L_{K(1)}S^2$.

Theorem (W.)

There exists an essential $\rho_n : S\langle\det_\pm\rangle \to R_n$ which is invertible in $\pi_\bigstar R_n$. Further, the action of $\mathbb{Z}_p^\times$ on the summand

$$\mathbb{Z}_p\{\rho_n^j\} \subseteq [S\langle\det_\pm\rangle^{\otimes j}, R_n]$$

is by j^{th} power of identity character.

Related work of Eric Peterson gives a more algebro-geometric perspective.

Consequently, the same computation for $\pi_* L_{K(1)}S^0$ gives us:

Corollary

There exists a subgroup $\mathbb{Z}/p^{s+1} \subseteq [S\langle\det_\pm\rangle^{\otimes(p-1)p^s m}, L_{K(n)}S^1]$ for m coprime to p.

References

Original formulation:

- G.W. Whitehead:
  On the homotopy groups of spheres and rotation groups.
- M. F. Atiyah: *Thom complexes*.
- J.F. Adams: *On the groups $J(X)$, I – IV*.

Unit spectra:

- J. P. May, F. Quinn, N. Ray, J. Tornehave:
  E_∞ ring spaces and E_∞ ring spectra.
- M. Ando, A. Blumberg, D. Gepner, M. Hopkins, and C. Rezk:
  Units of ring spectra and Thom spectra.
- J. P. May: *What are E_∞ ring spaces good for?*.

Picard spectra:

- M. Ando, A. Blumberg, D. Gepner:
  Parametrized spectra, multiplicative Thom spectra, and the twisted Umkehr map.

References

Adams conjecture and algebraic K-theory of finite fields:

- D. Quillen: *The Adams Conjecture*, and *On the Cohomology and K-Theory of the General Linear Groups Over a Finite Field*.
- D. Sullivan: *Genetics of homotopy theory and the Adams conjecture*.
- E. Friedlander: *Fibrations in etale homotopy theory*.
- W. G. Dwyer: *Quillen's work on the Adams conjecture*.

Image of J space:

- J. P. May, F. Quinn, N. Ray, J. Tornehave: *E_∞ ring spaces and E_∞ ring spectra*.
- M. Mahowald: *The order of the image of the J-homomorphism*.
- H. R. Miller, D. Ravenel: *Mark Mahowald's work on the homotopy groups of spheres*.

References

Morava K and E-theory:

- J. Morava:
  Noetherian Localisations of Categories of Cobordism Comodules.
- P. Goerss, M. Hopkins:
  Moduli Spaces of Commutative Ring Spectra.
- E. Devinatz, M. Hopkins:
  Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups.
- M. Behrens, D. Davis:
  The homotopy fixed point spectra of profinite Galois extensions.

Determinantal K-theory:

- E. Peterson:
  Annular decomposition of coalgebraic formal variety spectra.
- C. Westerland:
  A higher chromatic analogue of the image of J.