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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Leanne Merrill **Email/Phone:** leannem@uoregon.edu / 518 461 7614

Speaker's Name: Bob Oliver

Talk Title: Local structure of groups and of their classifying spaces

Date: 1 / 30 / 14 **Time:** 9 : 30 am / pm (circle one)

List 6-12 key words for the talk: p-local structure, Martino-Priddy conjecture, fusion system, fusion category, classifying space, discrete p-toral group.

Please summarize the lecture in 5 or fewer sentances: The lecture defined the p-local structure of a group and connected this structure to the Bousfield-Kan localization of classifying spaces. It introduced the notions of fusion systems and fusion categories, and showed that they have many of the same homotopy theoretic properties. Finally, it discussed discrete p-toral groups and their fusion categories.

CHECK LIST

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Local structure of groups and of their classifying spaces

Bob Oliver

This will be a survey talk on the close relationship between the local structure of a finite group or compact Lie group and that of its classifying space. By the "p-local structure" of a group G, for a prime p, is meant the structure of a Sylow p-subgroup $S \leq G$ (a maximal p-toral subgroup if G is compact Lie), together with all G-conjugacy relations between elements and subgroups of S. By the p-local structure of the classifying space BG is meant the structure (homotopy properties) of its p-completion $BG^\wedge_p$.

For example, by a conjecture of Martino and Priddy, now a theorem, two finite groups G and H have equivalent p-local structures if and only if $BG^\wedge_p \simeq BH^\wedge_p$. This was used, in joint work with Broto and Møller, to prove a general theorem about local equivalences between finite Lie groups — a result for which no purely algebraic proof is known.

As another example, these ideas have allowed us to extend the family of p-completed classifying spaces of (finite or compact Lie) groups to a much larger family of spaces which have many of the same very nice homotopy theoretic properties.

Bob Oliver – <u>Local structure of groups and of their classifying spaces</u>

1/30/14

Let p be a prime.

The p-<u>local structure</u> of a finite group G:

- a Sylow p-subgroup S
- G-conjugacy relations among elements and subgroups of S.

For G, H finite groups, $S \in \mathrm{Syl}_p(G)$, $T \in \mathrm{Syl}_p(H)$,
G and H have the same p-local structure if:

there exists an isomorphism $S \xrightarrow[\cong]{\varphi} T$
such that if $P \le S$, $Q \le S$, $\varphi(P) \le T$, $\varphi(Q) \le T$
and $P \xrightarrow[\cong]{\alpha} Q$, $\varphi(P) \xrightarrow[\cong]{\varphi\alpha\varphi^{-1}} \varphi(Q)$
then: $\alpha \in \mathrm{Iso}_G(P,Q) \iff \varphi\alpha\varphi^{-1} \in \mathrm{Iso}_H(\varphi(P), \varphi(Q))$

conjugacy in G.

Notation: $G \sim_p H$.

Two spaces X, Y have the same p-local structure if $X^\wedge_p \simeq Y^\wedge_p$ (the Bousfield-Kan p-completion).

Note: If X is p-good, (for example, if $|\pi_1 X| < \infty$) then $X^\wedge_p \simeq Y^\wedge_p$ if and only if there exist $X \xrightarrow{f} Z \xleftarrow{g} Y$ such that f, g are isomorphic in $H_*(-; \mathbb{F}_p)$.

<u>Martino-Priddy conjecture</u> (now a theorem):
For all G, H finite groups, $BG^\wedge_p \simeq BH^\wedge_p \iff G \sim_p H$

The $\Rightarrow$ direction is proven by Martino–Priddy, based on work of Mislin, consequence of the Sullivan conjecture.

The $\Leftarrow$ direction: only known proofs depend on the classification of finite simple groups.

Think of $\Leftarrow$ direction as a refinement of:

<u>Theorem</u> (Cartan, Eilenberg):

$$H^*(G; \mathbb{F}_p) = \varprojlim_{\substack{p\text{-subgroups} \\ \text{and conjugation.}}} H^*(-; \mathbb{F}_p)$$

Note that $BG^\wedge_p \simeq BH^\wedge_p \Rightarrow H^*(BG, \mathbb{F}_p) \cong H^*(BH; \mathbb{F}_p)$.

<u>Application</u>:

<u>Theorem</u> (Broto – Møller – Oliver):

Assume $\mathbb{G}$ is a connective reductive group scheme over $\mathbb{Z}$ (eg: GL_n, SL_n, Sp_{2n}, E_n, ...)

Suppose p is a prime and q, q' prime powers such that $p \nmid q q'$. Then

$$\mathbb{G}(\mathbb{F}_q) \sim_p \mathbb{G}(\mathbb{F}_{q'}) \text{ if}$$

$$\overline{\langle q \rangle} = \overline{\langle q' \rangle} \leq \mathbb{Z}_p^\times.$$

Remark: When $p = 2$, $\overline{\langle q \rangle} = \overline{\langle q' \rangle} \Leftrightarrow q \equiv q' \bmod 8$
and: $v_2(q^2 - 1)$
$= v_2((q')^2 - 1)$.

The only known proof of the above statement is not group theoretic (even though the statement of the theorem is).

Fusion systems:

- encodes the p-local information in a group.

Let G be a finite group, $S \leq G$ a Sylow subgroup. Let $\mathcal{F}_S(G)$ be the category whose objects are the set of all subgroups: $\mathrm{Ob} = \{P \leq S\}$ and morphisms are:

$$\mathrm{Mor}_{\mathcal{F}_S(G)}(P, Q) = \mathrm{Hom}_G(P, Q)$$

$$= \{\varphi : P \to Q \mid \varphi = c_g \text{ for some } g \in G\}.$$

Thus: for $S \leq G$, $T \leq H$ as before,

$G \sim_p H \iff \exists\, \varphi : S \xrightarrow{\cong} T$ which induces an isomorphism $\mathcal{F}_S(G) \xrightarrow{\cong} \mathcal{F}_T(H)$.

This leads to:

Definition: For a finite p-group S, a fusion system over S is a category $\mathcal{F}$ where

$\mathrm{Ob}(\mathcal{F}) = \{P \leq S\}$ all subgroups.

and $\forall P, Q$: $\mathrm{Mor}_{\mathcal{F}}(P, Q) \subseteq \mathrm{Inj}(P, Q)$

plus other axioms.

This definition is originally due to Puig. Technically, this is a saturated fusion system.

<u>Remark</u>: For a finite group G,

$$BG \simeq \underset{(G/p)}{\mathrm{hocolim}}\ BP.$$

<u>Want</u>: For any abstract fusion system $\mathcal{F}$, want to define a classifying space.

$$\underset{\overrightarrow{P \leq S}}{\mathrm{hocolim}}\ BP \quad .$$

The most naive thing to try is:

$$\underset{\overrightarrow{\mathcal{F}}}{\mathrm{hocolim}}\ B(-)$$

But this doesn't work (basically, too many morphisms).

<u>Definition</u>: Given a fusion system $\mathcal{F}$, over S, $P \leq S$ is $\mathcal{F}$-<u>centric</u> if

$$\forall\, P' \underset{\mathcal{F}}{\cong} P, \quad C_S(P') \leq P'.$$

We define a new category

$$\mathcal{O} := \mathcal{O}(\mathcal{F}^c) := \mathrm{Ob}(\mathcal{O}) = \left\{ \begin{array}{c} P \leq S \\ \mathcal{F}\text{-centric} \end{array} \right\}$$

$$\mathrm{Mor}_{\mathcal{O}}(P, Q) = \mathrm{Mor}_{\mathcal{F}}(P, Q) / \mathrm{Inn}(Q).$$

Thus, we have:

$$B : \mathcal{O}(\mathcal{F}^c) \to \underline{\mathrm{hoTop}}$$
$$P \longmapsto BP.$$

Then:

Definition: A classifying space for $\mathcal{F}$ is of the form:

$$B\mathcal{F} := \underset{\mathcal{O}(\mathcal{F}^c)}{\mathrm{hocolim}} (\hat{B})$$

for any $\hat{B} : \mathcal{O}(\mathcal{F}^c) \to \mathrm{Top}$ {rigidification of B}.

Dwyer-Kan: obstruction theory for rigidification.

Theorem (Chermak): For all $\mathcal{F}$, there exists a classifying space $B\mathcal{F}$, unique up to homotopy type.

Theorem: If $\mathcal{F} = \mathcal{F}_S(G)$, then $B\mathcal{F}^\wedge_p \simeq BG^\wedge_p$.

Homotopy properties of classifying spaces:

BG	$B\mathcal{F}$.
(Cartan-Eilenberg) $H^*(BG;\mathbb{F}_p) = \varprojlim H^*(-;\mathbb{F}_p)$	$H^*(B\mathcal{F};\mathbb{F}_p) \cong \varprojlim_{\mathcal{F}} H^*(B-;\mathbb{F}_p)$
For all p-groups Q, $[BQ, BG^\wedge_p] \cong \mathrm{Hom}(Q,G)/\mathrm{Inn}(G)$	$[BQ, B\mathcal{F}^\wedge_p] \cong \mathrm{Hom}(Q,S)/\sim$
$\mathrm{Out}(BG^\wedge_p) \subseteq [BG^\wedge_p, BG^\wedge_p]$ described by automorphism classes of its fusion.	$\underbrace{\mathrm{Out}(B\mathcal{F}^\wedge_p) \cong \left[\begin{matrix}\text{approximately} \\ \mathrm{Out}(\mathcal{F})\end{matrix}\right]}_{\text{Broto-Levi-Oliver}}$

Definition: A discrete p-toral group is an extension

$$1 \to (\mathbb{Z}/p^\infty)^n \to S \to \begin{pmatrix}\text{finite} \\ p\text{-group}\end{pmatrix} \to 1$$

$$\begin{array}{c} \cap\!\!\shortmid \\ T^r \end{array}$$

For all compact Lie groups G, there exists a maximal $S \leq G$, discrete p-toral, unique up to conjugacy.

Define $\mathcal{F}_S(G)$ as before.
Extend to abstract fusion systems over discrete p-toral groups.

Define $B\mathcal{F}$ as before.
Then $B(\mathcal{F}_S(G))^\wedge_p \simeq BG^\wedge_p$ for G a compact Lie group.

Also : p-compact groups, finite loop spaces.

You get the same homotopy properties, except:

$$H^*(B\mathcal{F}; \mathbb{F}_p) = ???$$

Thm (Levi-Libman): For all $\mathcal{F}$ over discrete p-toral groups there exists a unique $B\mathcal{F}$.
(extends Chermak's result).

Example: Maximal discrete p-toral subgroup in $O(2)$ at $p=2$:
$S = \langle 2\text{-power torsion in torus}, \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix} \rangle$.

More generally:

extension of torus by finite p-group.

For all

$$
\begin{tikzcd}
1 \arrow[r] & T \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\geq$}"] & \bar{S} \arrow[r] & \bar{S}/T \arrow[r] & 1. \\
& T_\infty & & &
\end{tikzcd}
$$

p-power torsion.

get extension using group cohomology.

it's equivalent to choosing a splitting:

$$
\begin{tikzcd}
1 \arrow[r] & T/T_\infty \arrow[r] & \bar{S}/T_\infty \arrow[r, shift left] & \bar{S}/T \arrow[l, dashed, shift left] \arrow[r] & 1.
\end{tikzcd}
$$

<u>Question</u>: Examples of abstract fusion system that isn't $\mathcal{F}_S(G)$ for some S, G?

<u>Answer</u>: Papers of Solomon in the '70s devoted to this, can construct examples at the prime 2.
Other primes have examples too, but the only way to know that they are exotic is to compare them to the classification of finite simple groups.

<u>Question</u>: Is there an analog of a "G-space" for an abstract fusion category?

<u>Answer</u>: Some work have been done: finding maps:

$$B\mathcal{F}_p^\wedge \longrightarrow BU(n)_p^\wedge$$