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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Leanne Merrill Email/Phone: leannem@uoregon.edu / 518 461 7614

Speaker's Name: Michael Hopkins

Talk Title: Equivariant Homotopy and Localization.

Date: 1 / 30 / 14 Time: 11 : 00 am / pm (circle one)

List 6-12 key words for the talk: equivariant cohomology, Segal-tom Dieck theorem, Mackey functor, equivariant additivity, representation sphere, Spanier-Whitehead duality

Please summarize the lecture in 5 or fewer sentances: This lecture introduced the main ideas in equivariant homotopy theory. It defined G-CW-complexes and showed how they differed from usual CW-complexes. It defined the category of equivariant G-spectra and the Burnside category and showed a fully faithful functor from the Burnside category to the homotopy category of G-spectra. It ended with a question about equivariant cohomology with coefficients in Mackey functors.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
    - **Computer Presentations**: Obtain a copy of their presentation
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- ☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
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Michael Hopkins - Equivariant Homotopy and localization 1/30/14

Space form problem: (see Manifold Atlas)

which manifolds have a sphere as a universal covering?

Answered (mostly) by Madsen, Thomas, Wall.

Tools that appear:

- Thom isomorphism
- Poincaré duality.

Equivariant Moore Space problem:

Can you realize groups in homology functorially? That is, given a group, can you find a space with that group in homology in a certain dimension, and no other homology?

Answer: (Carlsson) No.

Tools:

$H^*(BG; \mathbb{Z})$.

Steenrod algebra

Given $X \circlearrowleft G$, what can you say about

$$X^G = \{x \in X \mid gx = x \ \forall g \in G\}.$$

look at: fixed point formula.

Example: $G = T = \underbrace{S^1 \times \dots \times S^1}_{r \text{ times}}$, $X \circlearrowleft G$.

the equivariant cohomology of X $H^*_G(X)$ - module over $H^*_G(pt)$

$$H^*_G(pt) = \mathbb{Q}[x_1, \dots, x_r].$$

Theorem:

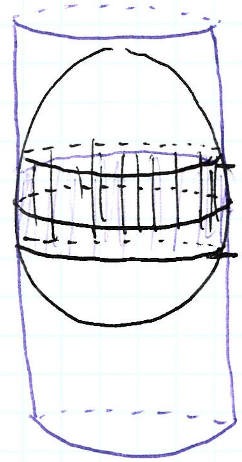
If $K := \mathbb{Q}(x_1, \ldots, x_r)$, then

$$K \underset{\mathbb{Q}[x_1,\ldots,x_r]}{\otimes} H^*_G(X) \approx K \otimes H^*(X^G).$$

Recommended paper: Atiyah-Bott,
The moment map and
equivariant cohomology.

A (relevant) theorem of Archimedes:

Given a sphere inscribed in a cylinder:



areas of the bands on the cylinder and on the sphere are equal.

can be proven using equivariant cohomology.

A philosophical point:

In homotopy theory, classifying "up to homotopy."

If $X \circlearrowleft G$ and $Y \circlearrowleft G$, then

equivariant → $\mathrm{Map}^G(X, Y) = \{f: X \to Y \mid f(gx) = gf(x)\}$.

Could ask:

Borel equivariant ↗ $\mathrm{Map}^G_\infty(X,Y) = \{f: X \to Y \mid f(gx) =_\infty gf(x)\}$.

The "$=_\infty$" means that it holds up to a contractible space of choices, in some sense.

Many theorems in equivariant theory are really meant in the Borel equivariant sense. But others, like the space form problem, use rigid equivariant theory.

We'll focus (mostly) on rigid equivariant theory.

For today: G a finite group.

Define a G-CW-complex, X. For $H \leq G$, X is built from $G/H \times D^n$, just like a CW complex.

<u>Remark</u>: If we have a ~~[illegible]~~ space with a trivial action, and a map from that space to X, it must land in the fixed points of X.

Thus: If X is a G-CW-complex
$\Rightarrow X^G$ CW-complex, and X is built from X^G by attaching cells $G/H \times D^n$ for $H \subsetneq G$.

What about products?

$$G/H \times D^n \times G/H' \times D^m$$

$$= S \times D^{n+m} \quad \text{where } S = G/H \times G/H'.$$

But this needs an equivariant cell decomposition!

Let $\mathcal{T}^G$ = (homotopy theory of) G-CW-complexes.

Now we stabilize:

Naive approach: work with spectrum objects.

That is, work in a category where homotopy classes of maps are calculated as follows:

$$[X, Y]_{st} = \lim_{n \to \infty} [S^n \wedge X, S^n \wedge Y]$$

(X a finite G-CW-complex).

$\mathcal{T}^G \longrightarrow$ "universal stabilization"

It's additive: $\vee X_\alpha \longrightarrow \prod X_\alpha$

and cofibration sequences = fibration sequences.

But: this category does not give Poincaré duality, or specifically:

Problem: Spanier-Whitehead duality doesn't work:

$G \circlearrowright K \hookrightarrow S^V$ where V is a representation of G,

S^V - one-point compactification.

["representation sphere"

So we replace the above definition with:

$$\{X, Y\}^G = \lim_{V \to \infty} [S^V \wedge X, S^V \wedge Y]^G$$

So we have a map:

$\mathcal{T}^G \longrightarrow \mathcal{S}^G$, the category of G-spectra.

$\mathcal{S}^G$ is the "right" category, but how to characterize the functor $\mathcal{T}^G \to \mathcal{S}^G$?
(meant as a framing question!).

Remember, we did this new stabilization to obtain Spanier-Whitehead duality.

We have:

- Finite G-sets are self-dual.

$$S_+ \wedge X \xrightarrow{\sim} \mathrm{Map}(S, X) \qquad S \text{ - a finite Gset.}$$

gives "additivity": $\bigvee_{s \in S} X \longrightarrow \prod_{s \in S} X$. (an equivariant analog).

But, in fact, this is equivalent.

Lesson: any time there is a monoidal structure, the group must act on the indexing set.

<u>Question</u>: What is $\{S^0, S^0\}^G$? Let $H \leq G$.

$$\lim_{V \to \infty} \mathrm{Map}(S^V, S^V)^G \longrightarrow \mathrm{Map}(S^{V^H}, S^{V^H}) = \mathbb{Z}$$

so we have a map:

$$\{S^0, S^0\}^G \longrightarrow \prod_{H \leq G} \mathbb{Z}.$$

So the "degree" of a map is different in the equivariant case.

Goal: A theorem of tom Dieck.

Question: what are maps from one finite G-set to another in this category?

Let Burn_G = objects: finite G-sets S.

Burnside category.

morphism: if
$$\begin{tikzcd} & U \arrow[dl] \arrow[dr] & \\ S & & T \end{tikzcd}$$
, take group completion.

ex $\mathrm{Burn}_G(pt, pt) = A(G)$

= Grothendieck group of finite G-sets.

Have a functor:

$$\mathrm{Burn}_G \longrightarrow \mathrm{ho}\mathcal{S}^G$$
$$S \longmapsto S_+$$

Theorem (Segal-tom Dieck) This is fully faithful.

$\mathrm{Burn}_G(S, T) = \{S_+, T_+\}^G$

is an isomorphism.

Sketch of proof:

By Spanier-Whitehead duality and reducing to orbits, it suffices to do $S = T = pt$.

want to show $A(G) \to \{S^0, S^0\}^G$ is an isomorphism.

Can show
$$\begin{tikzcd} A(G) \arrow[r] \arrow[dr] & \{S^0, S^0\}^G \arrow[d] \\ & \prod_{H \leq G} \mathbb{Z} \end{tikzcd}$$
$T \longmapsto \# T^H$

(diagonal map is a monomorphism).

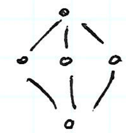
Need to show it is an epimorphism.

Let T = finite G-set.

$T^G = \phi$.

$T^H \neq \phi$. for all $H \subsetneq G$.

$$T_+ \longrightarrow S^0 \longrightarrow \tilde{T}$$



$\tilde{T}^G = S^0$

$\tilde{T}^H = \vee S^1$.

Have:

$$\begin{tikzcd}
\{S^0, T_+\}^G \arrow[r] & \{S^0, S^0\}^G \arrow[r, "*"] & \{S^0, \hat{T}\}^G \\
\mathrm{Burn}_G(pt, T) \arrow[u, "\approx"] \arrow[r] & A(G) \arrow[u] \arrow[r] & \mathbb{Z} \arrow[u, phantom, sloped, "\cong"]
\end{tikzcd}$$

$$S \longmapsto \# S^G$$

— conclusion of the proof.

Look at:

$$\{S^0, \hat{T}\}^G = \lim_{V \to \infty} [S^V, \tilde{T} \wedge S^V]^G$$

$$\begin{tikzcd}
S^{V^G} \arrow[r] \arrow[dr] & \tilde{T} \wedge S^V \\
& S^0 \wedge S^{V^G} \arrow[u]
\end{tikzcd}$$

What happens we attach another cell?

$$G/H \times S^n \longrightarrow \tilde{T} \wedge S^V$$

$n \leq \dim V^H$

$$\Updownarrow$$

$$S^n \longrightarrow (\tilde{T} \wedge S^V)^H = \vee S^1 \wedge S^{V^H}$$

So map $*$ above is surjective.

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Have a connection:

finite G-sets $\longleftrightarrow$ equivariant additivity

- introduced: slice tower, an analog of Postnikov tower.

another connection:

If $X \in \mathcal{S}^G$, look at $\pi_*^G X$.

then

$$[S_+, X]^G : \mathrm{Burn}_G \to \mathrm{Abel}.$$

is a Mackey functor.

An interesting problem:

What can you say about

$$H_G^*(S^V, M)$$

where M is a Mackey functor.

(work on Kervaire invariant problem is based on small cases of this.)

More on the connection with additivity:

$$C^J \longrightarrow C$$

for equivariant case, have to set up both sides separately (rather than in the usual case, where they can be derived.).