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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Leanne Merrill **Email/Phone:** leannem@uoregon.edu / 5184617614

Speaker's Name: Nitya Kitchloo

Talk Title: Homotopy Theory of Kac-Moody Groups

Date: 1 / 30 / 14 **Time:** 2 : 00 am / pm (circle one)

List 6-12 key words for the talk: Weyl group, Kac-Moody group, infinite dihedral group, Schubert basis, Lie group, stable homotopy theory.

Please summarize the lecture in 5 or fewer sentances: This lecture started by giving some simple examples of Kac Moody groups ~~[illegible]~~ and their weyl subgroups. It then provided a general definition of Kac-Moody groups in dimensions higher than 2. It discussed interesting cohomological and homotopical properties of Kac-Moody groups and their classifying spaces.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☑ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Nitya Kitchloo – Homotopy Theory of Kac-Moody groups 1/30/14

G – compact Lie group that is simple, connected, simply connected.

example: $G = SU(n)$

Let $T \subseteq G$ be the maximal torus. of rank n.

example: If $G = SU(n)$, $T = \Delta$-matrices (rank $n-1$).

The Weyl group $W = N_G(T)/T \circlearrowright T$.

- W acts on $\pi_1(T)$.
- W: a finite group generated by n-reflections.

In fact, G can be recovered from the W-action on $\pi_1(T)$.

Rank 2 example:

Let $a, b > 0$ be positive integers.
Let $W(a,b)$ be the group generated by the reflections $r_1 = \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}$; $r_2 = \begin{pmatrix} 1 & 0 \\ a & -1 \end{pmatrix}$.

Any group generated by two reflections is a dihedral group. Which one is this?

To see if $W(a,b)$ is (in)finite, we need to look at $(r_1 r_2)^k$.

claim: $(r_1 r_2)^k = \begin{pmatrix} d_{2k+1} & -d_{2k} \\ c_{2k} & -c_{2k-1} \end{pmatrix}$

Where c_i, d_i are defined as:

$$c_0 = d_0 = 0.$$
$$c_1 = d_1 = 1.$$
$$c_{i+1} = a d_i - c_{i-1}$$
$$d_{i+1} = b c_i - d_{i-1}.$$

some examples: (take $c_i = d_i$ for simplicity)

c_i	$(a,b) = (1,1)$	$(2,2)$	$(3,3)$
	0	0	0
	1	1	1
	1	2	3
	0	3	8
	-1	4	21
	-1	5	55
	0	6	144
	1	7	377

$(2,2)$: growing linearly

$(3,3)$: even Fibonacci numbers; growing exponentially

If:

$a = b = 1, \quad (r_1 r_2)^3 = 1 \longleftrightarrow G = SU(3)$

$a = 2, b = 1, \quad (r_1 r_2)^4 = 1 \longleftrightarrow Spin(5)$

$a = 3, b = 1, \quad (r_1 r_2)^6 = 1 \longleftrightarrow G_2$

What about general (a,b)?

In all other cases, $|W(a,b)|$ is infinite.

<u>Motivating question</u>: If $|W(a,b)|$ is infinite, is there still a simply connected topological group $K(a,b)$ with Weyl group $W(a,b)$?

Answer: Yes!

We define: $K(a,b)$ is pushout (or amalgam, or colim)

$$\begin{tikzcd} T \arrow[r, "\begin{pmatrix} 0 & 1 \\ 1 & -\frac{b}{2} \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ -\frac{a}{2} & 1 \end{pmatrix}"'] \arrow[dr, Rightarrow] & S^1 \times SU(2) \arrow[d] \\ S^1 \times SU(2) \arrow[r] & K(a,b) \end{tikzcd}$$

if a,b even.

in the category of topological groups.

If either of a or b is odd, replace whichever matrix with another appropriate matrix (omitted.).

<u>What do we know about $K(a,b)$?</u>

$$H^{2k}(K(a,b);\mathbb{Z}) = H^{2k+3}(K(a,b);\mathbb{Z}) = \mathbb{Z}/(c_k,d_k)$$

the gcd of c_k, d_k.

Let p be a prime.
Then

exterior algebra.

$$H_*(K(a,b);\mathbb{F}_p) = E(x_3; x_{2m-1}) \otimes \mathbb{F}_p[x_{2m}]$$

degrees

where:

m = smallest integer k so that $p \mid (c_k, d_k)$.

What about (co)homology of homogeneous spaces?

It turns out:

$$H^{2k}\left(K(a,b)/T\,;\,\mathbb{Z}\right) = \mathbb{Z}\langle \delta_k \rangle \oplus \mathbb{Z}\langle \tau_k \rangle$$

for $k > 0$.

Schubert basis.

<u>More structure</u>:

$$\delta_i \cup \delta_j = c(i,j)\,\delta_{i+j}$$

where $c(i,j) = \dfrac{c_{i+j}\,c_{i+j-1}\cdots c_1}{c_i c_{i-1}\cdots c_1 c_j c_{j-1}\cdots c_1}$.

(a generalized binomial coefficient).

Each $c(i,j)$ is actually an integer!

Cohomology of classifying space:

$$\begin{array}{ccc}
H^*(BK(a,b) : \mathbb{F}_p) & = & \mathbb{F}_p[x_4 ; x_{2m}] \otimes E(x_{2m+1}) \\
\downarrow & & \downarrow \ (\text{for } p > 2) \\
H^*(BT ; \mathbb{F}_p) & \longrightarrow & H^*(BT ; \mathbb{F}_p)^{W(a,b)}
\end{array}$$

If $a = b = 2$,

$$H^*(BK(2,2) ; \mathbb{F}_p) = \mathbb{F}_p[x_4, .. x_{2p}] \otimes E(x_{2p+1})$$

m does <u>not</u> stabilize as you might expect! It depends on p.

In fact, $BK(2,2) = LBSU(2)\langle \beta \rangle$.

$$X_4 \longleftrightarrow ax^2 + by^2 - abxy \in H^4(BT; \mathbb{F}_p).$$

(it survives in H^4).

Remark: Rational cohomology is uninteresting:

$$H^*(BK(a,b); \mathbb{Q}) = \mathbb{Q}[X_4].$$

Finally, a construction of Kac-Moody groups.

(in reality, construct them in a different way, and show they are the same as this def.)

Generalization to higher rank:

Let $\mathfrak{h} = \bigoplus_{i=1}^{n} \mathbb{Z} h_i$.

n-reflections: $r_i(h_j) = h_j - a_{ji} h_i$; $a_{ii} = 2$.

Let $A = (a_{ij})$.

Make an assumption: Assume $a_{ij} = 0 \Longleftrightarrow a_{ji} = 0$.
and $a_{ij} \leq 0 \quad (i \neq j)$.

[Need these for relations in order to construct a Lie group. Unclear what these mean from the standpoint of homotopy theory.].

Fact: If $J \subseteq \{1, 2, \ldots, n\}$ such that $W(J) := \langle r_i, i \in J \rangle$ is finite, then it is the Weyl group of a compact Lie group $K(J)$. (J is allowed to be empty, and then $K(J)$ is a torus)

If $J = \phi$, $K(\phi) = \mathfrak{h} \otimes \mathbb{R}/\mathbb{Z}$.

so $K(\phi) \subset K(J)$ is a maximal torus $\forall J$.

Definition: Let $\mathcal{C}$ = poset of $J \subseteq \{1, 2, \ldots, n\}$ such that $|W(J)| < \infty$.

Define $K(A) = \operatorname{colim}_{\mathcal{C}} K(J)$

(in the category of topological groups)

Remark: These groups are always simply connected.

[in our previous example, our matrix was $\begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$. This generalizes.]

Properties of Kac-Moody groups $K(A)$:

- $W(A) = \langle r_i ; i \in \{1, 2, \ldots, n\} \rangle$ is the Weyl group of $K(A)$.

  in most cases, this is infinite. In fact, it's infinite unless the poset has a terminal element.

- $H^{2k}(K(A)/T ; \mathbb{Z})$ = free indexed on elements of $W(A)$ of length k

  ↳ still called a Schubert basis.

Can prove many theorems using nil Hecke algebra actions on this ring, such as:

- $H_*(K(A); \mathbb{F}_p)$ - finitely generated ring. (not true for cohomology in most cases!)
- $\exists\ E(x_{2i_1-1}, x_{2i_2-1}, \ldots, x_{2i_n-1}) \subseteq H_*(K(A); \mathbb{F}_p)$ an abelian Hopf subalgebra.

- $H_*(K(A);\mathbb{F}_p) \otimes_{\mathcal{H}} \mathbb{F}_p$ is evenly graded.

  where $\mathcal{H} := E(x_{2i_1-1}, x_{2i_2-1}, \ldots, x_{2i_n-1})$.

  This is the image of $H_*(K(A)) \subset H_*(K(A)/T)$.

Facts about $BK(A)$:

- $\operatorname{hocolim}_{\mathcal{C}} BK(J) \xrightarrow{\sim} BK(A)$

  (seems to follow from mysterious condition on matrices).

  This is miraculous because $K(A)$ is defined as a colimit in topological groups, not spaces. (it's not clear what happens if you relax the conditions on the matrices, but you may lose the Lie algebra connection).

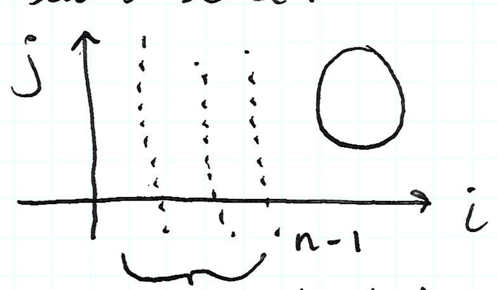
Also:

- $\operatorname{hocolim}_{\mathcal{C}} BW(J) \xrightarrow{\sim} BW(A)$.

Yields a Mayer Vietoris Spectral Sequence:

$$\varprojlim_{\mathcal{C}}{}^{i} H^j(BK(J)) \Rightarrow H^{i+j}(BK(A))$$

looks like:



concentrated here.

The spectral sequence shows:

- $H^*(BK(A); \mathbb{F}_p)$ is finitely generated as an algebra
- If $p > n+1$, then the spectral sequence collapses.
- If $p > n+1$, $H^*(BK(A); \mathbb{F}_p) \twoheadrightarrow H^*(BT; \mathbb{F}_p)^W$ and the Kernel = nilpotent elements
- (related to stable homotopy theory):

$$BN_K(T) \longrightarrow BK(A)$$

admits a stable splitting

<u>Note</u>: $K(A)/N(T)$ is not a finite complex, but it is acyclic if $p > n+1$.

(fiber of the map above.)

<u>Question</u>: classify all infinite reflection groups so that the hocolim is acyclic. (these are called "buildings.")