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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Leanne Merrill **Email/Phone:** leannem@uoregon.edu/518 461 7614

Speaker's Name: Tyler Lawson

Talk Title: Topological Automorphic Forms

Date: 1 / 30 / 14 **Time:** 3 : 30 am / pm (circle one)

List 6-12 key words for the talk: stable homotopy category, Adams-Novikov spectral sequence, formal group laws, complex bordisms, Morava K-theory, elliptic curves.

Please summarize the lecture in 5 or fewer sentances: This talk covered some naive ways to understand the stable homotopy category. It introduced formal group laws as a way to combine techniques from older theories. It then showed several examples where formal group laws gave invariants that did not appear previously. It ended with a preview of ways to generalize low-dimensional ideas to higher dimensions.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☑ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Tyler Lawson – Topological Automorphic Forms 1/30/14

Want to understand: $\mathcal{S}$ – stable homotopy category.

have an idea of cells in this category.

Can construct spectral sequences using cells:

① $\mathrm{Ext}_{\pi_*^s}(\pi_* X, \pi_* Y) \Rightarrow [X, Y]$.
(universal coefficients)

② $\mathrm{Tor}_{\pi_*^s}(\pi_* X, \pi_* Y) \Rightarrow \pi_*(X \wedge Y)$
(Künneth)

③ $H_*(X, \pi_*^s) \Rightarrow \pi_*^s(X)$
(Atiyah-Hirzebruch)

Disadvantages:

① Need $\pi_* X$.

$$\pi_*^s = \pi_* S^0 .$$

② π_*^s is an unfriendly ring.

③ Deceptive.

"$\mathcal{S}$ is like $Ch(\pi_*^s\text{-mod})$."

Main tool: Adams Spectral Sequence.

① $\mathrm{Ext}_{A^*}(H^*Y, H^*X) \Rightarrow [X, Y]$.

(A^* is the Steenrod algebra)

② $H^*(X) \hat{\otimes} H^*(Y)$

These are not only computational; they also give organizing principles and relations.

Good: ① organize things
② Allow computation.

Less Good: ① Hides information
② Doesn't "see" as many things.
(certain spectra are invisible).
③ Deceptive.

"$\mathcal{S}$ is like $\mathrm{ch}(A^*\text{-mod})$."
This is good - we understand the Steenrod algebra!

Adams - Nobikov Spectral sequence (ANSS):
Uses MU_*, MU_*MU (complex bordisms).
Then
$$\mathrm{Ext}_{MU_*MU}(MU_*X, MU_*Y) \Rightarrow [X,Y].$$

To understand the geometry, need to introduce:
Formal group laws: (FGL)
A FGL over a ring R is a power series
$$F(x,y) = x +_F y$$
$$= \sum a_{ij} x^i y^j$$
satisfying:

① $x +_F 0 = x$
② $x +_F y = y +_F x$
③ $(x +_F y) +_F z = x +_F (y +_F z)$.

Enters topology because of Chern classes:
$$c_1(\mathcal{L}) +_F c_1(\mathcal{L}') = c_1(\mathcal{L} \otimes_{\mathbb{C}} \mathcal{L}').$$

MU_* carries <u>the</u> <u>universal</u> FGL:

$MU_* MU$ parametrizes (strict) isomorphisms of FGLs.

$$f(x) = x + m_1 x^2 + m_2 x^3 + \cdots$$

$$\rightsquigarrow f^{-1}(f(x) +_F f(y)).$$

Given a spectrum R with appropriate multiplication,

① $MU_* R$ carries an FGL, and

② <u>operations</u> on $MU_* R$ come from a change-of-coordinates on the FGL.

Sometimes we <u>can</u> write down rings and FGLs (or diagrams of them) and realize them by spectra.

<u>Ex</u> $H\mathbb{Z} \longleftrightarrow \mathbb{G}_a$

$$F(x,y) = x + y.$$

<u>Ex</u> $K \longleftrightarrow \mathbb{G}_m$

sometimes is +.

$$F(x,y) = x + y - \beta xy$$

<u>Ex</u> Morava K-theory

over finite field $\mathbb{F}_q$, there are a lot of FGLs.

① height: $n \in \{1, 2, \ldots, \infty\}$.

② (Galois cohomology thing).

$\Rightarrow$ Morava K-theories (2-periodic versions)

Ex Universal deformations of FGLs over finite fields.

$$\mathbb{W}(\mathbb{F}_{q_0})[\![u_1, \dots, u_{n-1}]\!]$$

These realizes to the Morava E-theories (or Lubin-Tate spectrum, or Lubin-Tate theories, or E-theories), which have a coherently commutative multiplication.

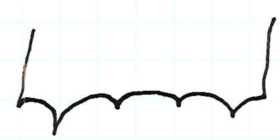
(This is due to Goerss - Hopkins - Miller).

These provides a dictionary between algebra and topology, and it is being forced by the topology!

These theories make us want more...

But, switching gears:

Automorphic forms often pictured like this:



But we want a different perspective.

Ex: Quadratics. Let R be a ring.

A quadratic is $x^2 + bx + c$.

A translation between them is a map

$$x \longmapsto x + t.$$

$$b \longmapsto b + 2t$$
$$c \longmapsto c + tb + t^2 .$$

Let $\frac{1}{2} \in R$.

we can complete the square and get a <u>complete</u> invariant $b^2 - 4c = \Delta$.

Let $2 = 0$ in R.

Then there is a new invariant:

$$b^2 = \Delta .$$

There is a new ring:

$\mathbb{Z}[b, c]$ parametrizes quadratics.

$\circlearrowright$ action by translations, Γ.

Then $\mathbb{Z}[b,c]^{\Gamma} = \mathbb{Z}[\Delta]$.

$$\begin{array}{ccc} \mathbb{Z}[b,c] & \twoheadrightarrow & \mathbb{Z}/2[b,c] \\ \big\downarrow \text{invariants} & & \\ \mathbb{Z}[\Delta] & \xrightarrow[\text{surjective}]{\text{not}} & \mathbb{Z}/2[b] \end{array}$$

$\hookrightarrow H^1$ (some cohomology).

There is <u>higher cohomology</u> for these quadratics.

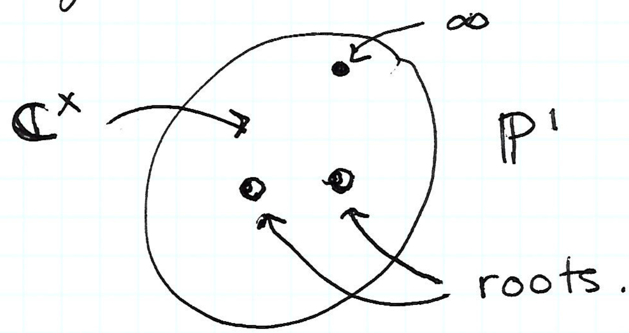
cohomology: $\mathbb{Z}[\Delta, \eta]/2\eta$.

$\hookrightarrow E_2^{ANSS}$ for $\pi_* ko$. why?

Given a quadratic, $\exists$ a formal group law:

$$x +_F y = \frac{x+y+bxy}{1-cxy}.$$

usually we can take out the roots:



This is the connection to homotopy theory.

<u>Ex</u> Elliptic curves over R:

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6.$$

Translations: $y \longmapsto y + rx + s$

$x \longmapsto x + t$.

In the same way: <u>action</u> on $\mathbb{Z}[a_1, \ldots, a_6]$.

<u>Invariants</u>: (calculated by Deligne)
(more information in Silverman's book)

$$\mathbb{Z}\left[c_4, c_6, \frac{c_4^3 - c_6^2}{1728}\right]$$

elliptic discriminant Δ.

There is also higher cohomology here:

mod 2: a_1 is invariant.
mod 3: $(a_2 - a_1^2)$ is invariant.

} More information in Tillman-Bauer.

Elliptic curves have multiplication rules, with an identity:

$$u = \frac{-y}{x},$$

expand multiplication for elliptic curves in terms of u.

The cohomology: E_2^{ANSS} for computing $\pi_* \mathrm{tmf}$.

<u>Ex</u> Formal group laws.

MU_* universal ring of formal group laws.

change-of-variables using power series.

$\Longrightarrow E_2^{ANSS}$ for computing π_*^S.

All of these algebraic examples gave us interesting topological information.

They have the advantage of having "sane" multiplication (as opposed to that of power series). Usually they involve rational functions, which are more practical coordinates for computation.

Quadratics: Detect information from heights ≤ 1.

Elliptic curves: Detect information from heights ≤ 2.

Algebraic geometry uses K3 surfaces, heights ≤ 10. (Szymik)

(So far, the formal group laws we have seen in this talk only go up to dimension 1).

In order to find FGLs of large height, we need to go further (work of Artin-Schreier):

$$y^{p-1} = x^p - x.$$

$\circlearrowleft$

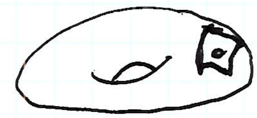
curves, which have Jacobians, abelian schemes.

Group action on Artin-Schreier curves

$\downarrow$

Actions on Jacobians

$\downarrow$



Action on formal group expansions near the identity.

The group action can slice out a 1-dimensional FGL from the large one.

- Why doesn't this come from the elliptic curve?
  - Comes from the fact that the group algebra map factors through the p-adic completion, which has more idempotents.

- Why not just think of random formal group laws?
  - Families that arise from the technique above have more tractable properties.