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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Leanne Merrill **Email/Phone:** leannem@uoregon.edu / 518 461 7614

Speaker's Name: Thomas Church

Talk Title: Representation Stability and Applications to Homological Stability.

Date: 1 / 31 / 14 **Time:** 9 : 30 am / pm (circle one)

List 6-12 key words for the talk: Configuration space, representation of symmetric group, combinatorial stability, FI-modules, FI-groups, congruence subgroups

Please summarize the lecture in 5 or fewer sentances: This lecture introduced the notion of multiplicative stability in configuration space. It then connected this notion to that of square-free polynomials and congruence subgroups, which have different notions of stability. It then connected these notions using homological algebra and representation theory.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
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- ☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
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Thomas Church - <u>Representation Stability and Applications to Homological Stability</u>. 1/31/14

<u>Outline</u>: Act I: Configuration spaces and multiplicity stability.

Act II: Square-free polynomials and combinatorial stability.

Act III: Congruence subgroups and inductive stability.

Act IV: FI-modules and finite generation

Act V: FI-groups and uniform generating sets

Act VI: Unifying homological stability and representation stability.

Joint work with:

C = Church	2010:	[CF]
E = Ellenberg	2011:	[C]
F = Farb	2012:	[CEF]
N = Nagpal		[CEFN]
P = Putman	2013:	[CP]
		[CEF_2] (second paper)
	2014?:	[CE]
		[C_2].

Act I:

Configuration Space: $\mathrm{Conf}_n(M)$ = space of n-element subsets $S \subset M$.

Ex: $\mathrm{Conf}_n(\mathbb{C}) = \{S \subseteq \mathbb{C} \mid |S| = n\}$
$= K(\mathrm{Braid}, 1)$.

Theorem (Arnold, Cohen):
$$H_*(\mathrm{Conf}_n(\mathbb{C}); \mathbb{Z}) \xrightarrow{\simeq} H_*(\mathrm{Conf}_{n+1}(\mathbb{C}); \mathbb{Z})$$
for $n \geq 2*$

and $\varinjlim H_*(\mathrm{Conf}_n(\mathbb{C}); \mathbb{Z}) = H_*(\mathrm{Maps}(S^2, S^2); \mathbb{Z})$

and $H_*(\mathrm{Conf}_n(\mathbb{C}); \mathbb{Q}) = \begin{cases} \mathbb{Q} & * = 0 \\ \mathbb{Q} & * = 1 \\ 0 & * \geq 2. \end{cases}$

Theorem (McDuff, Segal):
For any open M, $H_*(\mathrm{Conf}_n(M); \mathbb{Z}) \xrightarrow{\simeq} H_n(\mathrm{Conf}_{n+1}(M); \mathbb{Z})$.
for $n \gg *$.

Why not closed M?
1) $\mathrm{Conf}_n(M) \not\longrightarrow \mathrm{Conf}_{n+1}(M)$.
2) False: e.g. $H_1(\mathrm{Conf}_n(S^2); \mathbb{Z}) = \mathbb{Z}/(2n-2)\mathbb{Z}$

(closed: Interior of a compact manifold with bdry).

Theorem: (C): For any M,
$$H_*(\mathrm{Conf}_n(M); \mathbb{Q}) \simeq H_*(\mathrm{Conf}_{n+1}(M); \mathbb{Q})$$
for $n > *$.

By transfer: $H^*(\mathrm{Conf}_n(M); \mathbb{Q}) = H^*(\widetilde{\mathrm{Conf}}_n(M); \mathbb{Q})^{S_n}$
where $\widetilde{\mathrm{Conf}}_n(M) = M^n - \{z_i = z_j\}$.

Reduced to understanding S_n-invariants inside S_n-representations $H^*(\widetilde{\mathrm{Conf}}_n(M); \mathbb{Q})$

Now we do have maps $\widetilde{\mathrm{Conf}}_{n+1} \to \widetilde{\mathrm{Conf}}_n$!

Theorem (C.):

Decomposition of $H^i(\widetilde{\mathrm{Conf}}_n(M); \mathbb{Q})$ is stable for $n \geq 4i$.

What do we mean by stable? See chart. (later).
Idea: Each representation that occurs shows up in exactly the pattern pictured.

Act II: (throughout, all polynomials monic).

How many degree 10 polynomials in $\mathbb{F}_3[t]$?

$3^{10} = 59{,}049$.

How many of these are square-free?

$= 39{,}366$. Note: $\frac{39{,}366}{59{,}049} = \frac{2}{3}$.

How many do we expect to have distinct linear factors? That is, what is the average number of linear factors?

$$\frac{29{,}526}{39{,}366} = 75.0039\%. \quad \left(\text{very close to } \tfrac{3}{4}\right).$$

Definition: Conf_n = space of square-free degree n polynomials.

Ex: $\mathrm{Conf}_2 = \{T^2 + bT + c \mid b^2 - 4c \neq 0\}$

$\mathrm{Conf}_{10}(\mathbb{F}_3) = \{T^{10} + a_1 T^9 + \dots + a_{10} \mid \overset{a_i \in \mathbb{F}_3}{\Delta} \neq 0\}$.

(can use any field in this way; definition of discriminant is dependent upon this choice.)

It is consistent with previous notation:

$$\mathrm{Conf}_n\,\mathbb{C}(\mathbb{C}) = \{\text{square-free polynomials in } \mathbb{C}[T]\}$$

$$(T-\lambda_1)\cdots(T-\lambda_n)$$

$$\uparrow \qquad\qquad \updownarrow ? \quad \text{Yes.}$$

$$(\lambda_1, \ldots, \lambda_n) \quad \{S \subset \mathbb{C} \mid |S| = n\}.$$

Grothendieck-Lefschetz:

$$|\mathrm{Conf}_n(\mathbb{F}_q)| = q^n \sum (-1)^i q^{-i} \dim H^i(\mathrm{Conf}_n(\mathbb{C}); \mathbb{Q})$$

$$= q^n - q^{n-1}.$$

Statistically:

$$P(X_1, X_2, \ldots) \rightsquigarrow \chi_P : \mathrm{Conf}(\mathbb{F}_q) \longrightarrow \mathbb{Q}$$

$$\chi_P(f) = P(\#\text{linear factor, quadratic}, \ldots).$$

or:

$$\rightsquigarrow \chi_P : S_n \to \mathbb{Q}$$

$$\chi_P(\sigma) = P(\#\,1\text{-cycles}, 2\text{-cycles}, \ldots).$$

But, using Grothendieck-Lefschetz:

$$\sum_{f \in \mathrm{Conf}_n(\mathbb{F}_q)} \chi_P(f) = q^n \sum (-1)^i q^{-i} \langle H^i(\mathrm{Conf}_n(\mathbb{C}); \mathbb{Q}), \chi_P \rangle_{S_n}.$$

Reduces stabilization of statistics to:

Theorem (CEF_2): combinatorial stability.

For any $P(X_1, X_2, \ldots)$

$$\langle H^i(\widetilde{\mathrm{Conf}}_n(\mathbb{C}); \mathbb{Q}), \chi_P \rangle \text{ is constant}$$
$$\text{for } n \geq 2i + \deg P.$$

The inner product is the inner product of class functions.

Have correspondence:

linear stable range $\longleftrightarrow$ power-savings for $O(N^{\frac{1}{2}+\varepsilon})$, $N \simeq q^n$.

Act III:

Another example of homological stability:

Theorem (Charney):

$$H_*(SL_n\mathbb{Z}; \mathbb{Z}) \longleftrightarrow H_*(SL_{n+1}\mathbb{Z}; \mathbb{Z})$$
$$\text{for } n \geq 3*.$$

Fails for congruence subgroups:

$$\Gamma_n(p) = \ker(SL_n\mathbb{Z} \to SL_n\mathbb{Z}/p).$$

Ex: $H_1(\Gamma_n(p); \mathbb{Z}) = \mathfrak{sl}_n\mathbb{Z}/p$.

$$\Gamma_n(p) \to I + pA \longrightarrow A \bmod p.$$

Fails to be stable; but if we take into account action of deck group ($SL_n\mathbb{Z}/p$), it can be repaired.

<u>Def</u> given $T \subset \{1, \ldots, n\}$.

$$SL_T \mathbb{Z} = \left\{ M \in SL_n \mathbb{Z} \mid M_{ij} = \delta_{ij} \text{ if: } \begin{array}{l} i \notin T \text{ or} \\ j \notin T \end{array} \right\}$$

Called "inductive stability."

<u>ex</u>: If $T = \{1, 2\}$,

$$SL_T \mathbb{Z} = \begin{pmatrix} * & 0 \\ 0 & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{pmatrix}$$

Let $\Gamma_T(p) = (SL_T \mathbb{Z}) \cap (\Gamma_n(p))$.

<u>Theorem</u> (after Putman, CE, CEFN).

For $n \geq 2^{i-1}$,

$$H_i(\Gamma_n(p); \mathbb{Z}) = \operatorname*{colim}_{T \subsetneq [n]} H_i(\Gamma_T(p); \mathbb{Z}).$$

(Used by Caligari + Emerson, Caligari in number theory).

Techniques for proof? That is Act IV.

Act IV:

<u>Definition</u>: <u>FI</u> = category whose objects are finite sets and whose morphisms are injections.

<u>FI-module</u>: A functor $V : FI \to R\text{-mod}$.

<u>Ex</u> $T \longmapsto SL_T \mathbb{Z}$ makes $SL_\circ \mathbb{Z}$ into an FI-group.

$T \longmapsto \Gamma_T(p)$ makes $\Gamma_\circ(p)$ into an FI-group.

Then $H_i(\Gamma_\circ(p); \mathbb{Z})$ is an FI-module.

Then: $\widetilde{\mathrm{Conf}}_T(M) = \mathrm{Inj}(T, M)$

$(T \longmapsto \widetilde{\mathrm{Conf}}_T(M))$, $(FI^{op} \to \text{Spaces})$

so $H^i(\widetilde{\mathrm{Conf}}_n(M); \mathbb{Z})$ is an FI-module.

<u>Definition</u> (not ideal!)

An FI-module V is finitely generated (f.g.) if $\exists\, v_1, \ldots, v_n$ that "span" V.

<u>Theorem</u> (CEF):

Let V be a FI-module over $\mathbb{Q}$ (with some extra assumptions).

Then the following are equivalent:

1) V is finitely generated.
2) "multiplicity stability" for S_n-reps. V_n.
3) combinatorial stability for $\langle V_n, \chi_p \rangle_{S_n}$.

<u>Theorem</u> (CE, CEFN) Let V be an FI-module over any R.

Then the following are equivalent:

1) V is finitely presented.
2) inductive stability.

Note: $V_n = \operatorname{colim}_{T \subsetneq [n]} V_T$.

Theorem (CEFN) FI-modules are Noetherian if R is Noetherian.
and if V is finitely generated then $W \subseteq V$ implies W is finitely generated.

Act V:

$\mathrm{Aut}(F_n)$: What is an analog of congruence? free group. k-step nilpotent quotient

$$\mathrm{Aut}_n[k] = \ker\left(\mathrm{Aut}(F_n) \to \mathrm{Aut}(N_n^k)\right)$$

Theorem (CP):
$\mathrm{Aut}_n[k]$ is generated by elements supported on uniformly small splittings

$$F_n = A * B$$

where $\varphi(A) = A$,
$\varphi|_B = \mathrm{id}$.

Have exact sequence:

$$1 \to \mathrm{Aut}_n[1] \to \mathrm{Aut}^+(F_n) \to SL_n\mathbb{Z} \to 1.$$

$\mathrm{Aut}_n[1] \supseteq$ {normal generators} $\longleftrightarrow$ {relations in presentation} $\subseteq SL_n\mathbb{Z}$

Need a presentation of $SL_n\mathbb{Z}$:

$$SL_n\mathbb{Z} = \left\langle E_{ij} \;\middle|\; \begin{array}{l}[E_{ij}, E_{jk}] = E_{ik} \\ [E_{ij}, E_{kl}] = 1\end{array},\ (E_{12}E_{21}^{}E_{12}^{-1})^4 = 1\right\rangle$$

Note: All relations lie in some $SL_T\mathbb{Z}$ with $|T| \leq 4$.

and : $SL_n \mathbb{Z} = \underset{\substack{T \subsetneq [n] \\ |T| \leq 4}}{\mathrm{colim}} \; SL_T \mathbb{Z}$.

(for $SL_n R \leftrightarrow$ stable range for unstable $K_2(R)$).

Act VI :

$G = \Sigma$, GL, (Braid, surfs).

Have structure : $\amalg$, $\oplus$, ...

In general, write : $\amalg : G \times G \to G$.

$\rightsquigarrow$ G-Mod : $[G, Ab]$ symmetric, monoidal.

Can make sense of G-Algebras : $X = \{1\}$, $X = \mathbb{R}$.

- $T = T(X)$ free algebra on $\mathbb{Z}\,\mathrm{Hom}_c(X, -)$ (Yoneda embedding)
- $S = S(X)$ free commutative algebra on $\mathbb{Z}\,\mathrm{Hom}_c(X, -)$.
- A = constant functor $\underline{\mathbb{Z}}$.

Theorem (C_2) :

1) $\mathrm{Tor}_i^T(V, N_\circ) \approx 0 \iff$ twisted stability for $H_*(GL_n\mathbb{Z}; V_n)$

2) $\mathrm{Tor}_i^A(\underline{\mathbb{Z}}, \underline{\mathbb{Z}}) \approx 0 \Longleftarrow$ we can prove homological stability.

(this vanishing $\longleftrightarrow$ $\exists$ highly connected complex on which GL_n acts with nice stability).

Question: Is V polynomial at all?

Answer: S-mod $\leftrightarrow$ Dwyer central coefficient condition.

If we have this condition and V polynomial, can show
$\mathrm{Tor}^T_*(\mathbb{Z}, N_0) \Rightarrow \mathrm{Tor}^T_*(V, N_0)$.

Question: What is source category?

Answer: The functor is
$$- \otimes_T N_0 : T\text{-mod} \to N\text{-mod}.$$

What is a T-module? Tom means an algebra with a map $V \otimes T \to V$.

Can build bar resolutions:
$$\to A_+ \otimes A \to A \to A_0.$$
Can talk about Koszul properties, etc.

What does finitely presented FI-module mean?

Nothing has a projective resolution.

It means: V is f.g. if $\exists M$ such that $M \twoheadrightarrow V$ and M is a finite sum of representables.

V is finitely presented if $\exists K$ f.g., $0 \to K \to M \twoheadrightarrow V$.

Note: If S = free comm. algebra on S^1, get back symmetric spectra.

$H_2\big(\widetilde{\mathrm{Conf}}_n(\mathbb{C});\mathbb{Q}\big)$

$n = 3$	$\ydiagram{1}$
$n = 4$	$\ydiagram{1}^{\oplus 2} \oplus \ydiagram{1,1} \oplus \ydiagram{2}$
$n = 5$	$\ydiagram{1}^{\oplus 2} \oplus \ydiagram{1,1}^{\oplus 2} \oplus \ydiagram{2}^{\oplus 2} \oplus \ydiagram{2,1}$
$n = 6$	$\ydiagram{1}^{\oplus 2} \oplus \ydiagram{1,1}^{\oplus 2} \oplus \ydiagram{2}^{\oplus 2} \oplus \ydiagram{2,1}^{\oplus 2} \oplus \ydiagram{3}$
$n = 7$	$\ydiagram{1}^{\oplus 2} \oplus \ydiagram{1,1}^{\oplus 2} \oplus \ydiagram{2}^{\oplus 2} \oplus \ydiagram{2,1}^{\oplus 2} \oplus \ydiagram{3} \oplus \ydiagram{3,1}$
$n = 8$	$\ydiagram{1}^{\oplus 2} \oplus \ydiagram{1,1}^{\oplus 2} \oplus \ydiagram{2}^{\oplus 2} \oplus \ydiagram{2,1}^{\oplus 2} \oplus \ydiagram{3} \oplus \ydiagram{3,1}$

Stable answer for all $n \geq 7$

$n \geq 7$ $\ydiagram{1}^{\oplus 2} \oplus \ydiagram{1,1}^{\oplus 2} \oplus \ydiagram{2}^{\oplus 2} \oplus \ydiagram{2,1}^{\oplus 2} \oplus \ydiagram{3} \oplus \ydiagram{3,1}$