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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Leanne Merrill **Email/Phone:** leannem@uoregon.edu / 518 461 7614

Speaker's Name: Oscar-Randall Williams

Talk Title: Stability of modulispaces of manifolds

Date: 1 / 31 / 14 **Time:** 11 : 00 am / pm (circle one)

List 6-12 key words for the talk: Modulispaces, mapping class group, group homology, θ-structure, Borel construction, Cobordism group.

Please summarize the lecture in 5 or fewer sentances: This lecture introduced moduli spaces of manifolds and discussed their homological stability in certain cases. It surveyed several results in the field, and then showed part of a proof technique for homological stability, which involved the construction of a particular simplicial complex and a spectral sequence computation.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☑ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Oscar Randal-Williams: Stability of Moduli spaces of Manifolds

(1) Moduli spaces of manifolds:

a moduli space is anything where maps into it have a meaning.

Fix M a manifold (smooth) with boundary ∂M.

Let $B\mathrm{Diff}_\partial(M) =: \mathcal{M}(M)$.

and $\mathcal{E}(M) = \underbrace{E\mathrm{Diff}_\partial(M) \times_{\mathrm{Diff}_{\partial M}} M}_{\text{Borel construction}} \xrightarrow{\pi} \mathcal{M}(M)$.

<u>Note</u>: π is a fiber bundle with fiber M and structure group $\mathrm{Diff}_\partial(M)$.

It canonically contains

$$\mathcal{E}(M)_\partial = E\mathrm{Diff}(M) \times_{\mathrm{Diff}(M)} \partial M$$
$$= \mathcal{M}(M) \times \partial M.$$

Have a map: $[X, \mathcal{M}(M)] \to \left\{ \begin{matrix} E \\ \downarrow \\ X \end{matrix} \text{ smooth fiber bundle with fiber } M,\ E_\partial \cong X \times \partial M \right\}$

$$f \longmapsto \left(\begin{matrix} f^*\mathcal{E}(M) \\ \downarrow \\ X \end{matrix},\ (f^*\mathcal{E}(M))_\partial \cong X \times \partial M \right)$$

For reasonable X, the above map is a bijection.

$$\left(\longleftrightarrow * /\!/_{\mathrm{Diff}_\partial M} \right).$$

The ring $H^*(\mathcal{M}(M))$ is the ring of characteristic classes of such fiber bundles.

Hard to compute, but we should try! If $K : \partial M \to P$ is a cobordism, get a manifold $M \underset{\partial M}{\cup} K$.

Get a homomorphism:

$$\mathrm{Diff}_\partial(M) \longrightarrow \mathrm{Diff}_\partial(M \underset{\partial M}{\cup} K)$$

$$\Longrightarrow \quad \mathcal{M}(M) \longmapsto \mathcal{M}(M \underset{\partial M}{\cup} K).$$

This maps tend to induce isomorphisms in homology in certain ranges of degree.

But first:

① $M = \{1, 2, \ldots, n\}$.
Then $\mathrm{Diff}_\partial(M) = \Sigma_n$.
and $\mathcal{M}(\{1, 2, \ldots, n\}) \simeq B\Sigma_n$.

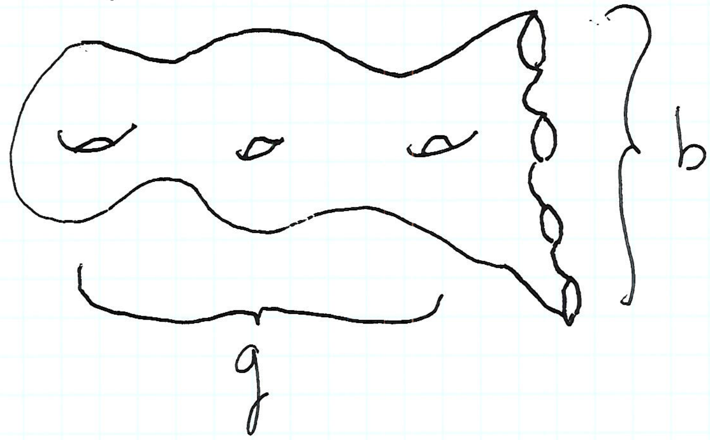
<u>Theorem</u> (Nakaoka):

$$H_i(B\Sigma_n) \longrightarrow H_i(B\Sigma_{n+1})$$

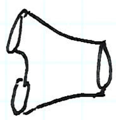
is an isomorphism for $i \leq \frac{n-2}{2}$ and an epimorphism for $i \leq \frac{n}{2}$.

(any coefficients).

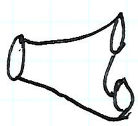
2 Let: $\Sigma_{g,b}$:



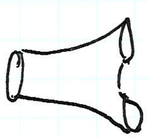
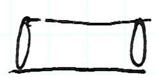
<u>Theorem</u> (Harer, Ivanov, Boldson, R-W):

—  $\mathcal{M}(\Sigma_{g,b}) \to \mathcal{M}(\Sigma_{g+1,b-1})$

is an epimorphism $3* \leq 2g+1$
and an isomorphism $3* \leq 2g-2$.

—  $\mathcal{M}(\Sigma_{g,b}) \longrightarrow \mathcal{M}(\Sigma_{g,b+1})$

is an isomorphism
for $3* \leq 2g$
and a monomorphism for all degrees.

—   $\simeq$  is an epimorphism for $3* \leq 2g+3$ and an isomorphism for $3* \leq 2g$.

—  : $\mathcal{M}(\Sigma_{g,b}) \to \mathcal{M}(\Sigma_{g,b-1})$
is an epimorphism in all degrees if $b-1>0$.

<u>Theorem</u> (Wahl, R-W):
same sort of thing happens for non-orientable surfaces.

Fix $\theta : B \longrightarrow BO(2)$ a fibration

$$\begin{tikzcd} \partial\Sigma \arrow[r, "l|_\partial"] \arrow[d] & B \arrow[d, "\theta"] \\ \Sigma \arrow[r, "\sigma"] \arrow[ur, dashed, "l"] & BO(2) \end{tikzcd}$$

surface Σ

l is a θ-structure.

Look at space of all θ-structures on Σ_s.

$$\Longrightarrow \mathcal{M}^\theta(\Sigma_{g,b}) = \left(\text{space of all } \theta\text{-structures on } \Sigma_{g,b}\right) /\!\!/ \mathrm{Diff}_\partial(\Sigma_{g,b})$$

<u>Theorem</u> (R-W):

$H_i(\mathcal{M}^\theta(\Sigma_{g,b}))$ stabilizes with g for all $i \geq 0$

if and only if

$H_0(\mathcal{M}^\theta(\Sigma_{g,b}))$ stabilizes with g.

$\theta : B\mathrm{Spin}(2) \to BO(2)$ studied by Haer, Tillman-Bauer.

$\theta : BO(2) \times X \to BO(2)$ trivial; studied by Cohen-Madsen ($\pi_1 X = 0$).

What invariants of θ does this depend upon?
What is relationship between BDiff and spin structure?

3 3-manifolds:

Hatcher, Wahl: study this phenomenon for mapping class groups.

They look at $\Gamma(M) = \pi_0 \mathrm{Diff}_\partial(M^3)$.

and show. $H_*(\Gamma(M \# P^{\# n}))$ stabilizes with n.

However, maps to $B\Gamma(M)$ don't mean much in this context.

≥ 6:

<u>Theorem</u> (Galatius - R-W):

Let $2n \geq 6$, let M be a 1-connected $2n$-manifold with nonempty boundary.

$- \cup ([0,1] \times \partial M \# S^n \times S^n) : \mathcal{M}(M) \to \mathcal{M}(M \# S^n \times S^n)$ induces an isomorphism in $* \leq \frac{g-3}{2}$ where $g = g(M)$ is $\max\{g \mid M \cong N' \#_g S^n \times S^n\}$

Does plumbing in the middle dimension stabilize? Sometimes (trivial normal bundle $\Rightarrow$ Yes).

<u>Theorem</u> (Perlmutter):

$$V_{g,1} = \#^g S^n \times S^{n+1} \setminus D^{2n+1}$$

$- \cup ([0,1] \times S^{n-1} \# S^n \times S^{n+1}) : \mathcal{M}(V_{g,1}) \to \mathcal{M}(V_{g+1,1})$.

is an isomorphism $* \leq \frac{g-3}{2}$ for $2n+1 \geq 9$.

Bergland-Madsen ~~[illegible]~~ calculate:

- $BhAut(\# S^n \times S^n \setminus D^{2n}) / \mathbb{Q}$
- $BDiff(\# S^n \times S^n \setminus D^{2n}) / \mathbb{Q}$.

How does one prove such a ~~[illegible]~~ theorem?

<u>Proof</u> of stability for Σ_n:

Think of a space on which the group acts, want it to be highly connected, and want stabilizers to be small.

Let S be a finite set.

Let $X_n = \mathrm{Inj}([n], S)$, a semisimplicial set.

complex of injective words.

Alternatively,

X_n = words of length $n+1$ in S, where each letter occurs ≤ 1.

Face maps: forget i^{th} letter of the word.

<u>Theorem</u> (Farmer '79): $|X_\bullet| \simeq \bigvee S^{|S|-1}$

$\iff |X_\bullet|$ is $(|S|-2)$-connected.

other proofs: Björner-Wachs '83.
Kerz '05
R-W '13.

<u>Clue</u> to the proof:

$|X_\bullet(S \setminus s)| \xrightarrow{f} |X_\bullet(S)| \to C_f$.

cofiber is a wedge of spheres.

also can use Hatcher Wahl coloring lemma.

We omit the difficult part (finding the complex).
We look at the purely formal part:

$S = [1, \dots, n]$. $\quad X_\circ = X_\circ(S)$.

$$\|X_\circ\| \longrightarrow \|X_\circ\| /\!\!/ \Sigma_n \xrightarrow{\varepsilon} B\Sigma_n.$$

ε ← augmentation map.

↑ an equivalence for our purposes.

So, compute homology of middle.

$$\|X_\circ\|/\!\!/\Sigma_n = \|X_\circ /\!\!/ \Sigma_n\| = \|[p] \longmapsto (\Sigma_n/\Sigma_{n-p-1} /\!\!/ \Sigma_n)\|$$

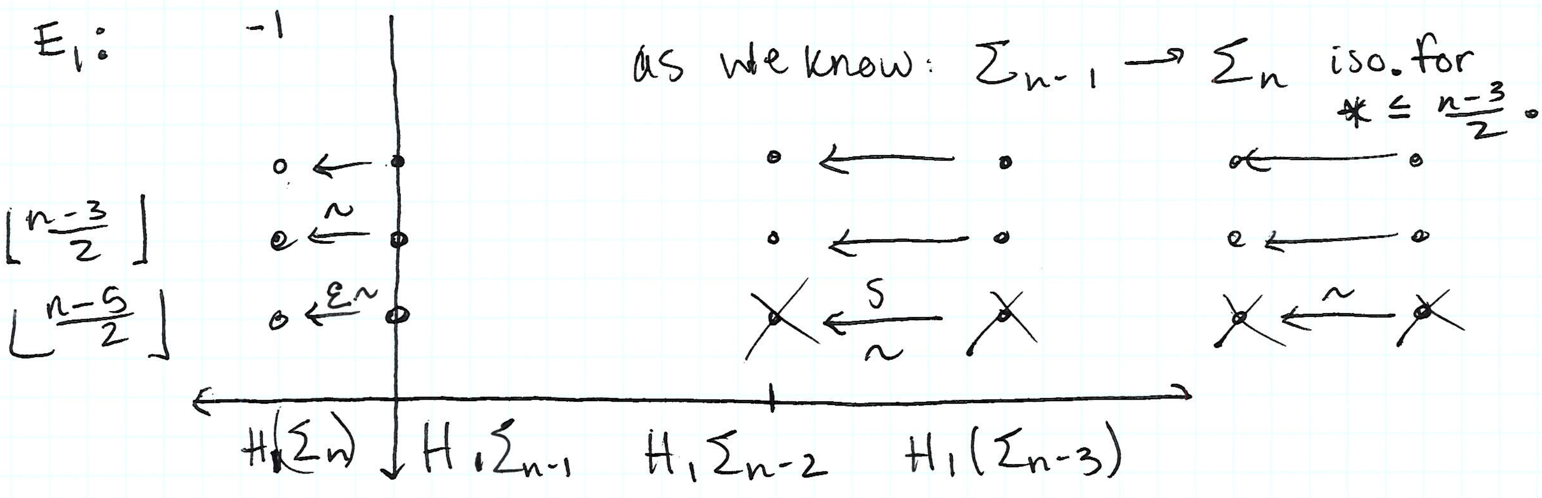
$$\Sigma_n/\Sigma_{n-p-1} /\!\!/ \Sigma_n = B\Sigma_{n-p-1}.$$

Also can see:

$d_i : B\Sigma_{n-p-1} \to B\Sigma_{n-p}$
are all freely homotopic.

Now find a spectral sequence with an extra column in -1: (for the cofiber):

as we know: $\Sigma_{n-1} \to \Sigma_n$ iso. for $* \le \frac{n-3}{2}$.



- Is simply connected essential?
  - Yes. (Soren's talk will explain more).
- How to detect surjectivity in cases when we do not have the symmetric group?
  - Harder.