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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Leanne Merrill Email/Phone: leannem@uoregon.edu/ 5184617614

Speaker's Name: Soren Galatius

Talk Title: Stable homology of moduli spaces of manifolds.

Date: 1 / 31 / 14 Time: 2 : 00 am / pm (circle one)

List 6-12 key words for the talk: Moduli spaces, scanning map, diffeomorphism group, configuration space, group completion, loop spaces.

Please summarize the lecture in 5 or fewer sentances: This talk introduced the idea of stable homology of groups and manifolds. It worked out some simple examples of scanning maps and gave a general definition. It then showed a typical proof that these maps induce isomorphism in a stable range in homology by introducing monoidal structures and group completion. Finally, it gave a higher-dimensional analog.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
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- ☑ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Soren Galatius - <u>Stable homology of moduli spaces of manifolds</u> 1/31/14

<u>Moduli spaces</u>: come in families indexed by $\mathbb{N}$:

$\mathcal{M}_n \quad n \in \mathbb{N}$:

<u>Ex</u>:
- $B\Sigma_n$
- $B\,\mathrm{Braid}_n$
- $B\,GL_n(\mathbb{Z})$ ← a bit outside these methods.
- $B\,\mathrm{Aut}(F_n)$
- $B\,\mathrm{Diff}(\Sigma_{n,1})$

$\vdots$

Exhibit homological stability in a certain range.

<u>Stable Homology</u>:

i) Often given by:

(iterated) loop spaces

ii) Often a map $\mathcal{M}_n \to \Omega(?)$ inducing an isomorphism in this stable range, called a "scanning" map.

<u>Other results</u>:

isomorphisms in H_* onto path component in stable range.

$B\mathrm{Br}_n \to \Omega^2 S^2$ Cohen, Segal.

$B\Sigma_n \to \Omega^\infty S^0$ Barratt-Priddy-Quillen-Segal

$B\mathrm{Diff}(\Sigma_{g,1}) \to \Omega^\infty(??)$ Tillman ~'95.

Madsen-Weiss, '02:

$$B\mathrm{Diff}(\Sigma_{g,1}) \to \Omega^\infty(\underbrace{MTSO(2)}_{})$$

Spectrum whose k^{th} space is $\frac{SO(n)}{SO(2)\times SO(n-2)} \wedge S^{n-2}$.

also, $\mathbb{C}P^\infty_{-1} = MTSO(2)$
$= BSO(2)^{-\gamma}$

A similar result: (G.–Randal-Williams):

$$BDiff\left(D^{2n} \# g(S^n \times S^n)\right) \to \Omega^\infty\left(BO(2n)[n+1,\infty)\right)^{-\gamma}$$

for $n > 2$.

How are these maps defined?

<u>Scanning map</u>:

<u>example</u>:

$$\begin{array}{ccc} B\,Br_n & \longrightarrow & \Omega^2 S^2 \\ \| & & \\ C_n(\mathbb{R}^2) & & \end{array}$$

Configuration space.

Let $A \subseteq \mathbb{R}^2$, $|A| = n$.

This is a point in $C_n(\mathbb{R}^2)$. Want: $\alpha(A) \in \Omega^2 S^2$.

How to define α?

Take $p \in \mathbb{R}^2$. Consider $\underbrace{(A - p) \cap B(0,\varepsilon)}_{\text{cardinality } 0 \text{ or } 1.} \subseteq B(0,\varepsilon)$.

So, $= B(0,\varepsilon) \cup \{\infty\}$ (one pt compactif.)

So consider a map from

$$\left\{\begin{array}{c}\text{one pt} \\ \text{cmptfication} \\ \text{of } \mathbb{R}^2\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{one pt} \\ \text{cmptfication} \\ \text{of } B(0,\varepsilon)\end{array}\right\}$$

which sends $\infty \longmapsto \infty$, and define

$\alpha(A) \in \Omega^2 S^2$ via this map.

Instead, could take: $\Psi =$ all discrete subsets $Q \subseteq \mathbb{R}^2$.

topologize: Q' near Q if it's a small perturbation inside a large ball.

Then, up to homotopy, the map can be described:

$$\begin{aligned} C_n(\mathbb{R}^2) &\longrightarrow \Omega^2 \Psi \\ A &\longmapsto \begin{pmatrix} p \in \mathbb{R}^2 \mapsto A - p \\ \infty \mapsto \infty \end{pmatrix} \end{aligned}$$

and $S^2 \xrightarrow{\simeq} \Psi$ is a homotopy equivalence.

- The same idea works for Σ_n, replacing $\mathbb{R}^2$ by $\mathbb{R}^\infty$
- with small modifications, works for diffeomorphism groups of manifolds.

On to higher dimensional manifolds:

<u>Oriented Surfaces</u>

Σ_1 a closed, orientable surface.

$$B\mathrm{Diff}^+(\Sigma_1) := \frac{\mathrm{Emb}(\Sigma_1, \mathbb{R}^\infty) \times \mathrm{Or}(\Sigma_1)}{\mathrm{Diff}(\Sigma_1)}.$$

orientation-preserving diffeos.

Could also have written (for consistency)

$$C_{\Sigma_1}(\mathbb{R}^\infty) = \{ Q \subset \mathbb{R}^\infty \mid Q \approx \Sigma_1 \}.$$

Approximated by $C_\Sigma(\mathbb{R}^n) = \dfrac{\mathrm{Emb}(\Sigma,\mathbb{R}^n) \wedge \mathrm{Or}(\Sigma)}{\mathrm{Diff}(\Sigma)}$

The scanning map is:

$$C_\Sigma(\mathbb{R}^n) \xrightarrow{\alpha} \Omega^n$$

$$\begin{array}{c} Q \\ \cap \\ \mathbb{R}^n \end{array} \longmapsto \left(\begin{array}{l} p\in\mathbb{R}^n \longmapsto \overbrace{(Q-p)\cap B(0,\varepsilon)}^{\text{empty or an affine subspace}} \\ \quad \infty \longmapsto \phi \\ \\ \end{array} \right)$$

Then:

$$\frac{SO(n)}{SO(2)} \underset{SO(n-2)}{\times} \mathbb{R}^{n-2} = \text{space of affine 2-dimensional oriented subspaces of } \mathbb{R}^n$$

and, one point compactified, it is:

$$\frac{SO(n)}{SO(2)_+} \underset{SO(n-2)}{\wedge} S^{n-2} = \text{space of affine 2-dimensional oriented subspaces of } \mathbb{R}^n \cup \{\infty\}.$$

$n \to \infty$:

$$B\mathrm{Diff}(\Sigma_{g,1}) \to B\mathrm{Diff}^+(\Sigma_g) \to \Omega^\infty MTSO(2).$$

The above is due to Madsen-Weiss.

F. Cantero: extended it for $5 \leq n < \infty$.

Wahl: unoriented surfaces.

Nariman: Diff^δ as discrete group.

Galatius: $\mathrm{Aut}(F_n)$.

How does one prove these maps are isomorphisms in a certain range? Reformulate slightly:

For example,

$$B\,Br_n \longrightarrow \Omega^2_n S^2 \quad \text{iso. in } H_*,\ * \leq \tfrac{n}{2}$$

$$\Updownarrow \quad \leftarrow H_*\text{-stable}$$

$$\mathbb{Z} \times B\,Br_\infty \longrightarrow \Omega^2 S^2 \quad \text{iso. in } H_* \text{ for all } *.$$

often a better way to prove these statements.

where $Br_\infty = \mathrm{colim}(Br_n \to Br_{n+1} \to \dots)$.

For example: $\underset{g\to\infty}{\mathrm{hocolim}}\ B\,\mathrm{Diff}(D^{2n} \# g(S^n \times S^n))$

$$\longrightarrow \Omega^\infty_0 BO(2n)[n+1,\infty)^{-\gamma}.$$

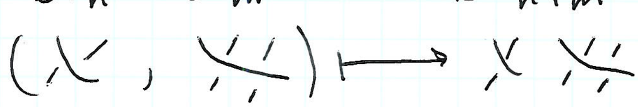
iso. in H_* for all n.

Monoid structures:

eg: $M = \coprod_{n\geq 0} B\,Br_n$

have a monoid structure given by putting braids next to one another:

$$Br_n \times Br_m \longrightarrow Br_{n+m}$$



gives a strictly associative monoid structure $M \times M \to M$.

can do similar things for $\coprod B\Sigma_n$, $\coprod B\,\mathrm{Diff}(\Sigma_{g,1})$

"Group-Completion" Theorem: (Quillen, McDuff-Segal).

Let $M = \coprod_{n \geq 0} M_n$ be a topological monoid, $\pi_0 = \mathbb{N}$,

which is homotopy commutative.

Then if $M_\infty = \operatorname{hocolim}(M_0 \xrightarrow{\cdot m} M_1 \xrightarrow{\cdot m} M_2 \to \ldots)$

we have:

$$\begin{tikzcd} M \arrow[rr] \arrow[dr] & & \Omega BM \\ & \mathbb{Z} \times M_\infty \arrow[ur] & \end{tikzcd}$$

is an H_*-isomorphism.

This implies:

$$\begin{array}{ccc} \mathbb{Z} \times BBr_\infty & \xrightarrow{H_*\text{-iso.}} & \Omega^2 S^2 \\ \Uparrow & & \\ B(\coprod BBr_n) & \xrightarrow{\simeq} & \Omega S^2 \end{array}.$$

These are general techniques that work in most of the previous examples.

How to prove the homotopy equivalence above?

Prove both sides $\simeq \{ A \in \Psi \mid A \subseteq \mathbb{R} \times (-1,1) \}$.

In general: (moduli spaces of higher dimensional manifolds)

- instead of a monoid, you have a category.
- often get "too big" moduli spaces
  eg. non-connected 2-manifolds.
- Need to get ride of those.

Point: The right hand side is (a priori) easier to understand than the left hand side.

E.g. $H^*(B\mathrm{Diff}(D^6 \# g\, S^3 \times S^3); \mathbb{Q})$
independent of g for $* \leq \frac{g-3}{2}$.

Theorem:

$$\mathbb{Q}[\varkappa_c \mid c \in B] \longrightarrow H^*(B\mathrm{Diff}(D^6 \# g\, S^3 \times S^3); \mathbb{Q})$$

is an isomorphism for $* \leq \frac{g-3}{2}$.

This indexing set is: $c \in B = \{ p_1^i e^j p_2^k \mid i,j,k \geq 0,\ |c| > 6 \}$.

$|\varkappa_c| = |c| - 6$ where p are Pontryagin classes and e are Euler classes.

The kappa classes are:

$$\begin{array}{c} W^6 \to E \\ \downarrow \\ B \end{array}, \quad c(T_* E) \in H^*(E).$$

What if manifold is not parallelizable?

Can look at:

$$\mathcal{M}^{\theta}(W^{2n} \# g\, S^n \times S^n) \longrightarrow \Omega^{\infty} X^{-\theta}$$

where:

$$\begin{tikzcd} & X \arrow[dr, "\theta"] & \\ W \arrow[ur] \arrow[rr, "TW"'] & & BO(2n) \end{tikzcd}$$

Moore Postnikov factorization

perform:

$$\frac{\text{"space of } \theta\text{-structures"}}{\text{Diff}}$$

construction.

What about manifolds with exotic bdry?

Doesn't matter; only depends on tangential n-skeleton.