MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Leanne Merrill Email/Phone: leannem@uoregon.edu/518 461 7614

Speaker's Name: Constantin Teleman

Talk Title: Loop Groups, TQFTs and algebraic geometry

Date: 1/31/14 Time: 3:30 am / pm (circle one)

List 6-12 key words for the talk: WZW, chiral WZW, modular functors, Chern-Simons theory, K theory, LG actions.

Please summarize the lecture in 5 or fewer sentances: The talk introduced the notion of WZW and chiral WZW, and talked about their connection to ~~[illegible]~~ Chern-Simons theory. It gave explicit constructions in algebraic geometry to understand the homology of stacks of bundles. Finally, it discussed recent results and open problems related to higher K-theories.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☑ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

1/31/14 ①

Constantin Teleman - Loop Groups, TQFTs, and algebraic geometry

Notes at https:/db.tt/w9lw46mu.

want to talk about:

- WZW CFT, relation to Chern-Simons modular functions in today's language.
- Algebraic geometry constructions
- what 'easy' topology sees (twisted K-theory)

what is WZW CFT?

WZW : 2-D Quantum field theory.

G - compact, simply connected Lie group.

"σ-model" with target G.

$k \in H^4(BG, \mathbb{Z})_{\geq 0} = \mathbb{Z}$, $\partial B = \Sigma$.

$g: \Sigma \to G$.

$$S(g) = \int_\Sigma \|g^{-1}dg\|^2 + \frac{1}{8}\int_B \langle g^{-1}dg, [g^{-1}dg, g^{-1}dg]\rangle$$

$$\int_{g:\Sigma \to G} e^{ikS(g)} = Z^{k,G}_{WZW}.$$

holonomy of unitary germ $\longrightarrow G$.

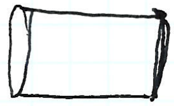
classical solution factors:

$g(z) = u(z)v(\bar{z})$, u, v holomorphic to $G_{\mathbb{C}}$.

This makes us hope that quantum theory is (holomorphic) × antiholomorphic.

space of states:

$$\mathbb{H}_{S^1} = L^2(LG;\ \mathcal{O}(K\Theta)) \oplus H_i \otimes H_i^\vee$$



(the irreducible (projective) "positive energy" representations of LG).

germ: $K \in H^4(BG) \longrightarrow \Omega^2 K \in H^2(LG)$.
(trangresses)

defines $\mathcal{O}(K\Theta) \to LG$.

Fact: comes from $S^1 \hookrightarrow \widetilde{LG}$

$$\begin{array}{c} \widetilde{LG} \\ \downarrow \\ LG \end{array}$$

If G simple, $\widetilde{LG}$ determined from K.

$S^1 \circlearrowright LG$ automorphisms.

Ask: extend to representations of $S^1 \ltimes \widetilde{LG}$

Ex: $G \circlearrowright V$, LG LV
$\hookrightarrow \Lambda^*(L^+V) \otimes \Lambda^*(L^{\leq 0}V)^\vee$.

Segal's Modular Functors:

A modular functor is
- A 3D tqft, Chern-Simon (CS)
- A holomorphic boundary condition.

(more info: Relative Quantum Field Theory).

A QFT is: a functor

$$CS: (\text{Bordism category}) \xrightarrow{CS} (\text{some } \mathbb{C}\text{-linear higher category})$$

$\mathbb{1} \searrow$ (#'s vector spaces linear categories) $\swarrow \alpha$

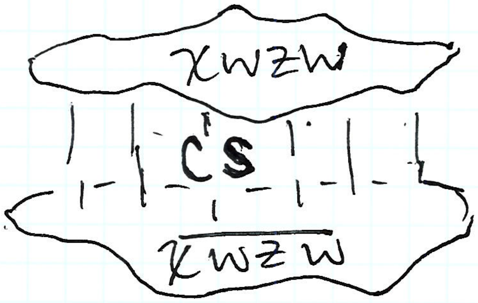
α is a natural transformation that will be our boundary condition (it may be sensitive).

Manifold	**CS theory**	$\xrightarrow{\chi WZW \text{ a "chiral WZW"}}$	$\mathbb{1}$
S^1	linear category (modular tensor category) $=$ PE Reps (LG, k) $\mathcal{R}(LG, k)$	$\xrightarrow[\text{vector space functor.}]{\text{underlying}}$	Vect.
$\Sigma, (\partial\Sigma = \phi)$	$V(\Sigma, G, k)$ "conformal blocks" (has structure of Frobenius algebra)	$\xrightarrow[\text{in } V]{\text{covector}}$	$\mathbb{C}$
$(\Sigma, \partial\Sigma)$ with $\partial\Sigma$ out	object in $R \in \mathcal{R}^{\otimes \partial\Sigma}$	$\xrightarrow[\chi WZW(R)]{\text{covector in}}$	$\underbrace{\otimes\, \mathbb{C}}_{\text{as a functor Vect}\to\text{Vect.}}$

all of the above varies holomorphically as you vary the complex structure on the surface.

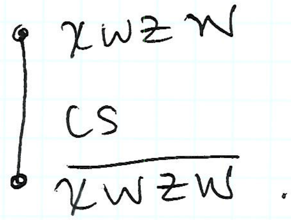
Special property of Chern-Simons theory: Unitarity.
- Unitarity of WZW:
  1. unitarity of CS theory.
  2. $(\chi WZW)^{\vee} = (\overline{\chi WZW})$

<u>Prop</u>: WZW is a sandwich of CS between χWZW and $\overline{\chi\text{WZW}}$.



$$\begin{tikzcd} \mathbb{1} \\ CS \arrow[u] \\ \mathbb{1} \arrow[u] \arrow[uu, bend right] \end{tikzcd}$$

= Reduction of CS along



$$WZW(X) = CS(X \times I).$$

<u>Question</u>: Does it matter that CS has a framing anomaly? Should there be a four-manifold?

<u>Answer</u>: No, not in this case.

<u>Question</u>: What is central charge for χWZW? How does it vary with rank or level?

<u>Answer</u>: It varies with the rank.

WZW : Maps$(\Sigma; G)$

CS : Maps$(M; BG)$ and G-bundles.

<u>Algebraic Geometry Constructions</u>: $\partial\Sigma = \phi$.

$H^*_{dR}(\mathrm{Map}(\Sigma; BG); \mathcal{O}(K(\Theta)))$ is the naive guess, $\quad K \in H^4(BG)$, $\Sigma \backslash K \in H^2(\mathrm{Map}(\Sigma; BG))$

In fact, need coherent sheaf cohomology (not deRham).

- $\mathrm{Bun}_G(\Sigma)$ stack of alg. $G_{\mathbb{C}}$ bundles
- $\mathcal{O}(k\Theta)$ complex structure unique if G is simple.

Then

$$V(\Sigma, G, k) = H^0(\mathrm{Bun}_G(\Sigma); \mathcal{O}(k\Theta))$$

(Hitchin, Beilinson - K.):

projectively flat connection in here over M_g.

$\chi_{WZW} : H^0 \to \mathbb{C}$: evaluation at trivial bundle.

not horizontal in this case.

<u>Open problem</u>: Construct Hermitian metric on the H^0 bundle.

Next one:

$(\Sigma, \partial\Sigma) \rightsquigarrow H^*(\mathrm{Map}(\Sigma; \partial\Sigma; BG) : \mathcal{O}(k\Theta))$

LG acts on these spaces projectively.

$\mathrm{Bun}_G(\Sigma^d, \hat{p})$ $\quad p = \langle p_1, \ldots, p_n \rangle$.

$\uparrow \Sigma^d$

$\mathcal{O}(k\Theta)$ algebraic

<u>Theorem</u> $H^0(\mathrm{Bun}_G(\Sigma; \hat{p}) : \mathcal{O}(k\Theta))$
is a finite mult. positive energy rep.
of $\mathrm{Map}(\Sigma; G)$
and $= R \in \mathrm{Ob}(\mathcal{R}(LG; k))$.

<u>Q</u>: Did someone show that these satisfying gluing axioms? Yes.
How much of this can we detect using standard topological methods?

<u>Index Theorem</u>: X compact, complex.
If $\mathcal{G} \curvearrowright X \leftarrow \mathcal{L}$ (line bundle).

$\mathcal{G}$ acts $\Sigma(-1)^q H^q(X)$

$[\mathcal{L}] \in K^0_{\mathcal{G}}(X) \xrightarrow{p_!} K^0_{\mathcal{G}}(x) = R(G)$

Then $p_![\mathcal{L}] = \Sigma(-1)^q H^q(X;$

If G acts projectively: $\tau \in H^2(BG; \hat{\mathbb{C}}_{ch})$.

$[\mathcal{L}] \in {}^{\tau}K_G(X : \mathbb{Z}) \xrightarrow{p_!} {}^{\tau}K^0_G(pt)$.

Want: $p_! \; {}^{\tau}K_{LG}(X) \to {}^{\tau}K_{LG}(pt)$

Would like to get K-cocycles representing R.

${}^{\tau}\cancel{K_{LG}}(pt)$ ${}^{\tau}K_{LG}(\mathcal{A})$ ← classifying line bundle for proper LG actions

doesn't make sense as stated

$\mathcal{A} = \Omega^1(S^1; \mathfrak{g})$, LG acts by gauge transformations.

$\mathcal{A} \to G$ holonomy map, G acts by conjugation.

${}^{\tau}K_{LG}(\mathcal{A}) \cong {}^{\tau}K_G(G)$

Theorem (Freed, Hopkins, T.).

(natural) isomorphism ${}^{\tau+c}K_0^{top}(G) \cong K(R(LG;k))$.

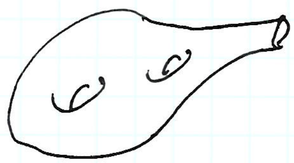
Fact: $\mathcal{A}_{flat}(\Sigma)/Map(\Sigma, \partial\Sigma; G)$

$$\begin{array}{c} \mathcal{A}_{flat}(\Sigma)/Map(\Sigma,\partial\Sigma;G) \cong Hol(\Sigma,\partial\Sigma). \\ \downarrow \\ \mathcal{A}_{S^1} \end{array}$$

restriction to boundary.

Finite dimensional model:

$$\begin{array}{c} Flat(\Sigma, \partial\Sigma). \\ \downarrow \\ G \circlearrowright Flat(S^1) \cong G. \end{array}$$



$$\begin{array}{c} G^{2g} \\ \downarrow \prod \text{commutators}. \\ G \end{array}$$

Theorem: $\pi_![1] \in {}^{\tau+c}K_0^{top}(G)$

"is" the K-class of R.

(over M_g'). [After inverting a bit, $k+c$].

Proven by (T., Woodward).

A new application: Extend techniques for line bundles to other vector bundles?

<u>Yes</u>: Replace with units in K-theory,

eg: $\sum q_6^n \, \mathrm{Sym}\, T\, \mathrm{Bun}_G \in K[\![q_6]\!]$.

Is there a way to deform CS theory to account for what we see in higher twisted K-theory?

Loop groups, alg geometry and TQFT

1. WZWCFT
2. Modular functors and boundary conditions
3. Alg geometry construction of some structure
4. Index formula in twisted K theory
5. K-linear TQFT from correspondences
6. Generalization, Higgs bundles
7. Open Questions

1. WZW is a 2D quantum field theory of maps into a compact Lie group G.

Parameter $k \in H^4(BG;\mathbb{Z})_{\geq 0}$;

Assume G simple, $\pi_1 G = 0$ for simplicity; $k \in \mathbb{Z}_{\geq 0}$

Action $k \cdot S(g)$ for a map $g : \Sigma \to G$, $\Sigma = \partial B$

$$S(g) = -\frac{1}{8\pi}\int_\Sigma \mathrm{Tr}_{\mathfrak{g}}(\gamma \wedge * \gamma) - \frac{1}{24\pi}\int_B \mathrm{Tr}_{\mathfrak{g}}(\gamma \wedge [\gamma,\gamma]), \quad \gamma = g^{-1}dg$$

Effect: classical solutions are $g = u(z)\overline{v(z)}$, u, v holomorphic to $G_{\mathbb{C}}$, instead of harmonic maps

$\Rightarrow$ Suggests a factorization of the quantum theory as holomorphic $\times$ antiholomorphic. This is true.

Example: Hilbert space of states $\mathbb{H} \subset$ "$L^2(LG; \mathcal{O}(k\Theta))$" where $\mathcal{O}(k\Theta) \to LG$ is the line bundle of the central extension $\widetilde{LG} \to LG$ defined by $\Omega^2 k \in H^2(LG;\mathbb{Z})$ factors as

$$\mathbb{H} \cong \bigoplus_i \mathcal{H}_i \otimes \overline{\mathcal{H}_i}$$

over the (k-projective, positive energy) irreps of LG and propagation happens holomorphically in each $\mathcal{H}_i$.

The combinatorics of how the theory on $\mathbb{H}$ assembles from the $\mathcal{H}_i$ was encoded by Segal in the concept of

2. Modular functors

In modern language: A modular functor is a 3D (1,2,3) Topological QFT plus a holomorphic boundary condition.

Spelt out a bit: a QFT is a symm. monoidal functor from a (geometric) bordism n-category to some $\mathbb{C}$-linear n-category having #'s, vect. spaces at the top.

Example: the trivial QFT (1, $\mathbb{C}$, Vect, etc). $\mathbb{1}$

A boundary condition for a QFT Q is a natural transformation $Q \xrightarrow{\alpha} \mathbb{1}$ (ignoring top mfolds).

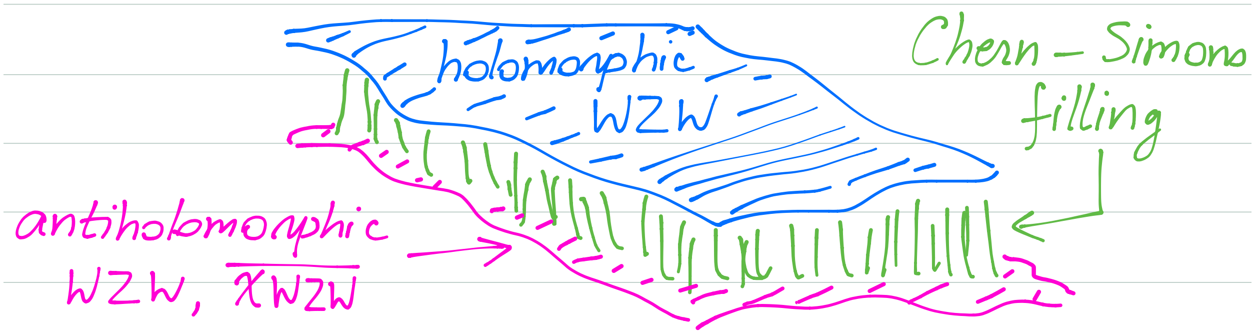
Holomorphic WZW is a boundary condition for a topological field theory (Chern-Simons); it is a gauge-fixing condition. (Placing it at the ends of an interval $\Rightarrow G$.)

Examples:

Manifold	Chern-Simons	chiral (X)WZW	$\mathbb{1}$
S^1	(modular tensor) category $\mathcal{R}(LG;k)$ of LG-reps, level k	$\xrightarrow[\text{underlying vector space}]{\text{representation}}$	Vect
Σ $(\partial\Sigma = \emptyset)$	Space $V(\Sigma;G,k)$ of "conformal blocks"	$\xrightarrow[V \to \mathbb{C}]{\text{covector}}$	$\mathbb{C}$
$(\Sigma, \underset{\text{out}}{\partial\Sigma})$	Object R in $\mathcal{R}^{\otimes \partial\Sigma}$	$\xrightarrow[\text{XWZW}(R)]{\text{covector in}}$	$\mathbb{C}$

2' Full WZW from the modular functor

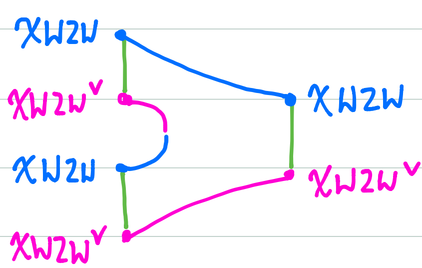
WZW is a sandwich of Chern-Simons theory between the holomorphic $\times$ antiholomorphic WZW boundaries



Reformulation: WZW is the dimensional reduction of Chern–Simons along the interval I, $\mathrm{WZW}(X) = \mathrm{CS}(X \times I)$

Remark 1. Unitarity of full WZW is captured by the isomorphism $\chi\mathrm{WZW}^\vee \cong \overline{\chi\mathrm{WZW}}$, in particular, there is a hermitian structure on conformal blocks. There is no known geometric construction of this structure.

Remark 2. The M.F. supplies an algebra structure on the full WZW conformal field theory, from the picture



"Fact 1" The algebra structure for WZW comes from the group structure on G.

3. Algebraic geometry construction of χWZW

(a) Conformal blocks $V(\Sigma; G; k)$, covector $\chi WZW : V \to \mathbb{C}$

V is part of CS, a gauge theory, quantizing maps to BG

Naively: take for V the de Rham $H^*(Map(\Sigma; BG); \mathcal{O}(k\Theta))$ where the line bundle $\mathcal{O}(k\Theta) \longrightarrow Map(\Sigma; BG)$ comes by transgression of $k \in H^4(BG)$.

However, $\mathcal{O}(k\Theta)$ is not flat (unless G is finite) but only holomorphic, for the incarnation of $Map(\Sigma; BG)$ as $Bun_G(\Sigma)$, the stack of all algebraic $G_{\mathbb{C}}$-bundles on Σ.

Def $V(\Sigma; G; k) = \Gamma(Bun_G(\Sigma); \mathcal{O}(k\Theta))$

$\chi WZW : V \longrightarrow \mathbb{C}$ is evaluation at trivial bundle

(b) Surface with boundary $(\Sigma, \partial\Sigma)$

Naive construction: (de Rham) cohomology of $\mathcal{O}(k\Theta)$ over $Map(\Sigma, \partial\Sigma; BG)$ = moduli of G-bundles on Σ, trivialized on $\partial\Sigma$. Carries action of $Map(\partial\Sigma; G)$.

As before, make algebraic: close up Σ to Σ^c with a subset P of marked points and let $Bun_G(\Sigma; \hat{P})$ be the stack of $G_{\mathbb{C}}$-bundles trivialized formally near P.

Theorem $\Gamma(Bun_G(\Sigma, \hat{P}); \mathcal{O}(k\Theta))$ is a finite type rep of $G^{\partial\Sigma}$.

As object in $\mathcal{R}(G,k)$, it is $V(\Sigma, \partial\Sigma; G, k)$.

$\chi WZW : V \longrightarrow \mathbb{C}$ is evaluation of sections at triv. bundle.

4. Topological aspects using (twisted) K-theory

If G acts on a complex $(\underset{\text{mfld}}{X}, \underset{\text{Line bdle}}{\mathcal{L}} \to X)$ then it acts on the $H^i_{\bar\partial}(X;\mathcal{L})$ and the virtual G-rep $\sum(-1)^q H^q$ is given by $p_![\mathcal{L}] \in K_G(*)$, $p_! : K_G(X) \to K_G(*)$ in equivar. K-thry

If G acts projectively on $\mathcal{L}$, classified by $\tau \in H^2(BG;\mathbb{C}^*)$ there are analogous ${}^{p^*\tau}K_G(X)$, ${}^{\tau}K_G(*)$ and $p_!$.

The multiplicities in $V(\Sigma, \partial\Sigma, G, k)$ should be captured by

$$p_! : {}^{p^*\tau}K_{LG}\left(\mathrm{Bun}_G(\Sigma, \hat{P})\right) \longrightarrow {}^{\tau}K_{LG}(*)$$

(Note: $k \in H^4(BG;\mathbb{Z}) \rightsquigarrow H^3(LBG;\mathbb{Z}) \cong H^3_G(G;\mathbb{Z}) \ni \tau$)
The nearest to ${}^{\tau}K_{LG}(*)$ in topology is ${}^{\tau}K_{LG}(\mathcal{A})$,
where $\mathcal{A}$ = classifying space for proper LG-actions
= $\mathfrak{g}$-connections on S^1 with gauge action of LG
$\Omega G \subset LG$ acts freely and so ${}^{\tau}K_{LG}(\mathcal{A}) \cong {}^{\tau}K_G(G)$.

Thm. [Freed-Hopkins-T] (1) ${}^{\tau+c}K_G(G) \cong K(\mathcal{R}(G,k))$.
(The shift c comes from the adjoint Spin rep. of LG)
(2) Have $\mathrm{HolBun}_G(\Sigma, \partial\Sigma) \cong \mathrm{Flat}(\Sigma, \partial\Sigma)$, with an LG-map p to $\mathcal{A}$, restriction of connection to $\partial\Sigma$.
Then, $K_{LG}(\mathrm{Flat}) \cong {}^{p^*(\tau+c)}K_{LG}(\mathrm{Flat})$; with that, $p_![1] \in {}^{\tau+c}K_{LG}(\mathcal{A}) \cong K(\mathcal{R}(G;k))$ is the class of $V(\Sigma, \partial\Sigma; G, k)$.

Explanation: $S^1 = \partial\Sigma \hookrightarrow \Sigma$

$$p : \mathrm{Flat}(\Sigma;\partial\Sigma) \longrightarrow \mathrm{Flat}(\partial\Sigma) = \mathcal{A}$$

$$\circlearrowleft LG \qquad\qquad\qquad \circlearrowleft LG$$

The subgroup $\Omega G \hookrightarrow LG$ acts freely and dividing it out gives the map

$$\pi : G^{2g(\Sigma)} \xrightarrow[\text{commutators}]{\text{product of}} G, \quad G\text{-equivariant}$$

$$\pi_! : {}^{\pi^*(\tau+c)}K_G(G^{2g}) \longrightarrow {}^{\tau+c}K_G(G) \cong K\mathcal{R}(G;k)$$

5. K-linear TQFT from ${}^{\tau+c}K_G(G)$

Chern-Simons theory admits a "softening" to a K-linear 2D TQFT.

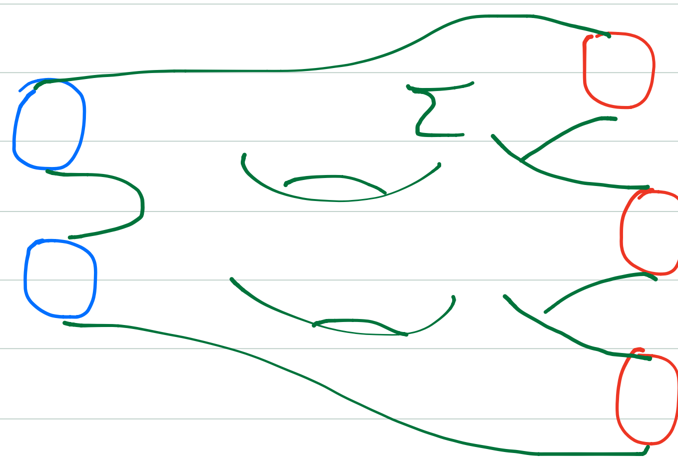
3d	numbers	gone
2d	vector spaces	points in K
1d	categories	K-modules
	functors	K-linear maps
(0d)	tensor cats	K-algebras

For example, the spaces $V(\Sigma; G, k)$ form an alg. vector bundle over $\mathcal{M}_g$, with projectively flat connection. The softening remembers its K-theory class

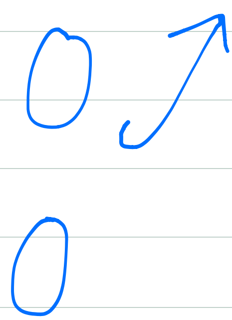
Theorem (Freed–Hopkins–T) "String topology correspondences" define a $\mathbb{K}$-linear 2D TQFT with space of states ${}^{\tau+c}K_G(G)$.
(T-Woodward) This agrees with the softening of CS theory, at least after inverting $(k+c)$. (Conjecturally, without).

Explanation:

$\partial^-\Sigma$



$\partial^+\Sigma$



leads to a correspondence diagram

$$\begin{tikzcd}[column sep=small] & {}^{\tau'}K(\mathcal{Fl}(\Sigma)) \arrow[dl, "p^+_!"'] \arrow[dr, "p^-_!"] & \\ {}^{\tau+c}K(\mathcal{Fl}(\partial^-\Sigma)) & & {}^{\tau+c}K(\mathcal{Fl}(\partial^+\Sigma)) \end{tikzcd}$$

$$ {}^{\tau+c}K(\mathcal{Fl}(\partial^-\Sigma)) \cong {}^{\tau+c}K_G(G)^{\otimes 2} \qquad {}^{\tau+c}K(\mathcal{Fl}(\partial^+\Sigma)) \cong {}^{\tau+c}K_G(G)^{\otimes 3}$$

$$\tau' = (p^+)^*(\tau+c) \cong (p^-)^*(\tau+c)$$

6. Generalization: Higgs bundles.

The topological methods that captured basic WZW information can be generalized to solve new problems (The gauged linear σ-model in Gromov-Witten theory).
The most natural example is the formula for the q-dimension

$$(*) \quad \sum_{n\geq 0} q^n H^0\left(Bun_G(\Sigma); \mathcal{O}(k\Theta)\otimes Sym^n T Bun_G(\Sigma)\right)$$

(This is the space of sections of $\mathcal{O}(k\Theta)$ over T^*Bun_G)
Remark The K-class over $\mathcal{M}_g$ is computable (not done)

Key observation: $\mathcal{O}(k\Theta)\otimes Sym_q T Bun_G$ is a unit in $K^0[[q]]$, a higher analogue of a line bundle
There are higher twistings of K^* going with that (Theory insufficiently developed in the equivariant case, although enough for our application).
The corresponding twisted $K_G(G)$, controlling $(*)$, can be computed explicitly and leads to

$$(*) = \left(\tfrac{1}{2}k+2\right)^{g-1}\sum_{m=1}^{k+1}\left(\sin\frac{m_q\pi}{k+2}\right)^{2-2g}\cdot\left(1+\frac{2q}{k+2}\,\frac{1-q\cos\frac{2\pi m_q}{k+2}}{(1+q)^2-2\cos\frac{2\pi m_q}{k+2}q}\right)^{g-1}$$

With $m_q = m + m_1 q + m_2 q^2 + \dots$ solving the equation

$$m_q + \frac{1}{2\pi i}\log\left(1-q\exp\left(\frac{-2\pi i\, m_q}{k+2}\right)\right) + (\text{cx. conj}) = m$$

7. Open questions

A. Compute ${}^{\tau}K_G(G)$ with higher twistings as TCFT (over $\mathcal{M}_g$ or $\overline{\mathcal{M}}_g$).
This relates to the gauged linear σ-model.

"Verlinde formula with ψ-classes"
Recent paper (Pandharipande et al) computes this for original Verlinde algebra. (Answer could also be derived from CFT).
In general, need a compactification of $Bun_G(\Sigma)$ over $\overline{\mathcal{M}}_g$ (nodal surfaces). Recent construction: Solis

B. Define ${}^{\tau}K_G(G)$ as a fully extended, K-linear TQFT.
Without the twisting and for simply connected G, this is "generated" by $K_G(*)$, whose HH_* is $K_G^*(G)$.
Attempting to generate from $K_*\langle G\rangle$ (w/ Pontryagin prod) fails and leads instead to $K_*(LBG)$.
(This gives a Moore-Segal style open-closed theory)
However, there is no known "${}^{\tau}K_G(*)$", $\tau \in H^4(BG;\mathbb{Z})$
This τ would define a K-gerbe over BG, or curving, and we are seeking the (K-linear) categ. of curved modules
This is like a mirror symmetric description of the TQFT but over K-theory!

C. Does the Higgs version of ${}^{\tau}K_G(G)$ come from a 3D TQFT (deforming Chern-Simons theory)?

At face value, the answer is NO; the modular tensor category does not deform ("Ocneanu rigidity", Etingof et al)

Deformation may be possible in more flexible settings:
- $\mathbb{Z}/2$ graded derived setting
  Quantum groups have more deformations there
- Algebras with more units than $\mathbb{C}$

A large class of C^*-algebras have recently been shown to have $GL_1(A) \sim \Omega GL_1(K\text{-theory})$ as spectra.

(Dadarlat - Pennig).