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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: James Freitag Email/Phone: freitag@math.berkeley.edu

Speaker's Name: Antoine Chambert-Lior

Talk Title: Hrushovski-Kazhdan's motivic Poisson formula and motivic height zeta functions

Date: 05/12/14 Time: 9:30 am / pm (circle one)

List 6-12 key words for the talk: Grothendieck ring, motivic integration, Motivic Poisson formula, valued fields, height function, model theory

Please summarize the lecture in 5 or fewer sentences: The Grothendieck ring of varieties is complicated and not well-understood. In order to prove results about the ring, one often uses motivic measures. In this talk, a motivic Poisson formula of Hrushovski-Kazhdan is used to prove the geometric analogue of a result Tschinkel.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Geometric height zeta functions and the motivic Poisson formula

Antoine Chambert-Loir (Université Paris-11)
Joint work with François Loeser (Université Paris-6)

arXiv:1302.2077

Model Theory in Arithmetic and Geometry
Berkeley, May 12-16, 2014

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Heights

Classically, the Height of a rational number is defined as

$$H(x) = \max(|a|, |b|), \qquad x = a/b, \quad a, b \in \mathbf{Z}, \quad \gcd(a, b) = 1.$$

It is some measure of its arithmetic complexity.
More generally, one defines the Height of a point $x \in \mathbf{P}^n(\mathbf{Q})$ by

$$H(x) = \max(|x_0|, \dots, |x_n|), \quad x = [x_0 : \dots : x_n], \; x_0, \dots, x_n \in \mathbf{Z}, \text{ coprime}.$$

Northcott's finiteness theorem:
For every $B \geqslant 0$, there are only finitely many $x \in \mathbf{P}^n(\mathbf{Q})$ with Height $\leqslant B$.

Schanuel's theorem: When $B \to \infty$, one has

$$\operatorname{Card}\left(\{x \in \mathbf{P}^n(\mathbf{Q}) \,;\, H(x) \leqslant B\}\right) \sim \frac{1}{\zeta(n+1)} 2^n B^{n+1}.$$

Height zeta function: Generating series

$$Z(s) = \sum_{x \in \mathbf{P}^n(\mathbf{Q})} H(x)^{-s}.$$

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HEIGHTS — MANIN'S QUESTION

GENERALIZATIONS:

- One can replace $\mathbf{Q}$ by an arbitrary number field F, or by a function field in one variable over a finite field;
- One can replace $\mathbf{P}^n$ by an arbitrary projective variety;
- One can vary the embedding and consider polarized varieties (X, L) — L is an ample line bundle on the projective variety X.

MANIN'S QUESTION: What can be said of the corresponding counting function $N_X(B)$? Does it have an asymptotic expansion:

$$N_X(B) \sim cB^s \log(B)^t.$$

What are the analytic properties of the associated generating series $Z_X(s)$? Abscissa of convergence? meromorphic continuation? poles?

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Manin's proposal

Manin proposed a GEOMETRIC INTERPRETATION which involves the geometry of the effective cone of the Néron-Severi group of X, and of the classes of L and K_X^{-1}. For example, when $L = K_X^{-1}$, he suggest that one has

$$s = 1 \quad \text{and} \quad t = \operatorname{rank}(\mathrm{NS}(X)) - 1.$$

Peyre refined this interpretation by interpreting the constant c as the volume of the adelic space $X(\mathbb{A}_F)$ for some "Tamagawa measure".

Necessary precautions:

- One assumes that the rational points of X are Zariski dense — the most important case is the one where X is a Fano variety and $L = K_X^{-1}$ is its anticanonical line bundle.
- It may be necessary to replace the ground field by a finite extension;
- It may be necessary to restrict the counting problem to a dense open subset.

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Theorems…

Manin–Peyre's expectation has been verified in a number of situations:

- Hypersurfaces of small degree compared to the dimension (circle method);
- Flag varieties and their generalizations (Franke–Manin–Tschinkel, using Langlands's theorems on Eisenstein series);
- Equivariant compactifications of some algebraic groups: tori (Batyrev–Tschinkel), vector spaces (Chambert-Loir–Tschinkel), simply connected semi-simple groups (Shalika–Takloo-Bighash–Tschinkel, Gorodnik–Maucourant–Oh), Heisenberg groups (Shalika–Tschinkel)…
- Del Pezzo surfaces (de la Bretèche, Browning, Derenthal, Le Boudec…).

The list is still growing.

… AND COUNTEREXAMPLES

There are also COUNTEREXAMPLES which show that it is sometimes necessary to avoid a *thin* subset.

- Total space of the universal family of diagonal cubic surfaces (Batyrev–Tschinkel);
- The Hilbert space of two points on $\mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1$ (Schmidt, Le Rudulier).

These counterexamples affect the finer aspects of the conjectural asymptotic expansion $N(B) \sim cB^s \log(B)^t$, namely the parameter t and the constant c, but not (not yet?) the parameter s.

Contents

Geometric heights

Let C be a projective curve over a field k and let $F = k(C)$.
Then there is a correspondence:

arithmetic over F	geometry over C
projective space $\mathbf{P}^n_F$	(trivial) fibration $\mathbf{P}^n_k \times C \to C$
point $x \in \mathbf{P}^n(F)$	morphism $\sigma_x \colon C \to \mathbf{P}^n_k$
height $h(x) = \log(H(x))$	degree $\deg(\sigma_x^* \mathcal{O}(1))$
polarized variety $(X, L)/F$	polarized fibration $(\mathcal{X} \to C, \mathcal{L})$
point $x \in X(F)$	section $\sigma_x \colon C \to \mathcal{X}$
height $h(x)$	degree $\deg(\sigma_x^* \mathcal{L})$

In the geometric context, Northcott's theorem becomes the statement that *sections σ of given degree d form a algebraic variety M_d over k.*
It is then natural to follow Peyre's suggestion (around 2000) to investigate how these varieties M_d vary with d.

Geometric height zeta functions

Let C be a projective curve over a field k and let $F = k(C)$. Let $\mathcal{X} \to C$ be a proper flat morphism, let $\mathcal{L}$ be a line bundle on $\mathcal{X}$. One lets $X = \mathcal{X}_F$, $L = \mathcal{L}|_X$ and assumes that L is *big*.

One adds a natural condition on points $x \in X(F)$/sections σ_x such as:
"x belongs to an appropriate subset U of X",
which insures that for every $d \in \mathbf{Z}$, sections $\sigma\colon C \to \mathcal{X}$ such that $\deg(\sigma^*\mathcal{L}) = d$ form a constructible set M_d, empty for $d \ll 0$.

Geometric height zeta function:

$$Z(T) = \sum_{d\in\mathbf{Z}} [M_d] T^d.$$

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Grothendieck groups of varieties

As a group, the Grothendieck ring KVar_k is defined by its *generators*:
isomorphism classes [X] of algebraic varieties X over k

subject to the *scissor relations*:
$[X] = [Y] + [X \setminus Y]$, whenever Y is a closed subscheme of X.

Its ring structure is given by:
$[X] \cdot [Y] = [X \times_k Y]$

This is a huge, complicated ring (non-noetherian, non-reduced,...) whose structure is not yet well understood.

Unit element: class of the point, $1 = [\mathrm{Spec}(k)]$

Lefschetz element: class of the affine line, $\mathrm{L} = [\mathbf{A}^1_k]$

Example: $[\mathbf{P}^n_k] = 1 + \mathrm{L} + \cdots + \mathrm{L}^n$.

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Motivic measures

One grasps properties of the Grothendieck ring KVar_k thanks to motivic measures which are functions μ from Var_k to a ring A such that

$$\mu(X) = \mu(Y) \qquad \text{if } X \text{ and } Y \text{ are isomorphic;}$$
$$\mu(X) = \mu(Y) + \mu(X \setminus Y) \qquad \text{if } Y \text{ is a closed subset of } X;$$
$$\mu(X \times_k Y) = \mu(X)\mu(Y).$$

Indeed these are exactly the ring morphisms from KVar_k to A.

One knows many interesting motivic measures:

- If k is finite, counting measure $X \mapsto X(k)$;
- If $k = \mathbf{C}$, Hodge-Deligne polynomial $X \mapsto E_X(u, v)$;
- Cohomological realizations into various other Grothendieck groups (Hodge structures, ℓ-adic representations, crystals...),
- and many other...

Example: The geometric Schanuel theorem

Let us consider the case of $\mathcal{X} = \mathbf{P}^n \times C$, polarized with $\mathcal{L} = \mathcal{O}(1)$. For simplicity, we assume that C has a k-rational point a.

Let M_d be the space of morphisms $\sigma\colon C \to \mathbf{P}^n_k$ such that $\deg(\sigma^*\mathcal{O}(1)) = d$. The height zeta function for $\mathbf{P}^n$ is the generating series

$$Z(T) = \sum_{d\geqslant 0} [M_d] T^d.$$

Using results of Kapranov, Peyre has shown the motivic analogue of Schanuel's theorem:

- $Z(T)$ is a rational power series;
- The power series $(1 - \mathrm{L}^{n+1}T)Z(T)$ converges at $T = \mathrm{L}^{-1}$ in a suitable completion $\widehat{\mathcal{M}}$ of the localized Grothendieck ring $\mathrm{KVar}_k[\mathrm{L}^{-1}]$;
- When $d \to \infty$, $\dim(M_d) - (n+1)d$ has a finite limit, and $\log(\kappa(M_d))/\log(d)$ converges to 0.

Here, $\kappa(M_d)$ is the number of irreducible components of M_d of maximal dimension.

Contents

Objects: the context

The situation is as follows:
Let C be an irreducible smooth projective curve over an algebraically closed field k of characteristic zero, let $F = k(C)$.

Let C_0 be a non-empty open subset of C.

Let $\mathcal{X}$ be a proper smooth variety endowed with a non-constant morphism to C, let $\mathcal{L}$ be a line bundle on $\mathcal{X}$, and let $\mathcal{U}$ be a Zariski open subset of $\mathcal{X}$.

Let $X = \mathcal{X}_F$, $U = \mathcal{U}_F$ and $L = \mathcal{L}|_X$ be the corresponding objects on the generic fiber. One makes the following geometric assumptions:

- U is isomorphic to the affine space $G = \mathbf{A}^n_F$, viewed as an algebraic group over F;
- X is a *smooth equivariant compactification of* G whose divisor at infinity $D = X \setminus G$ has strict normal crossings;
- $L = -(K_X + D)$.

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Objects: the moduli spaces

For every integer $d \in \mathbf{Z}$, we consider the "moduli space" M_d of sections $\sigma\colon C \to \mathcal{X}$ satisfying the following properties:

- $\deg(\sigma^*\mathcal{L}) = d$ — sections of geometric height d;
- $\sigma(C_0) \subset \mathcal{U}$.

The second condition means that we consider a variant for integral points of the geometric Manin problem. For this reason, the relevant line bundle is the log-anticanonical line bundle.

By general results on equivariant compactifications of affine spaces, the line bundle L is big and (some power) has a global section supported on D. This implies the following properties:

- The space M_d exists as a constructible set, and even as an algebraic variety over k if $\mathcal{L}$ is effective. In particular, it has a class $[M_d] \in \mathrm{KVar}_k$.
- M_d is empty for $d \ll 0$.

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OBJECTS: THE HEIGHT ZETA FUNCTION

Let $\mathcal{M}$ be the localization of KVar_k by the multiplicative subset of KVar_k generated by L and by the classes $\mathrm{L}^a - 1$, for $a > 0$.
We consider the generating Laurent series

$$Z(T) = \sum_{d\in\mathbf{Z}} [M_d] T^d \in \mathcal{M}[\![T]\!][T^{-1}]$$

with coefficients in this localization.

We introduce subrings

$$\mathcal{M}[T, T^{-1}] \subset \mathcal{M}\{T\}^\dagger \subset \mathcal{M}\{T\} \subset \mathcal{M}[\![T]\!][T^{-1}],$$

where $\mathcal{M}\{T\}$ is generated over $\mathcal{M}[T, T^{-1}]$ the inverses of the polynomials $1 - \mathrm{L}^a T^b$, for $b \geqslant a \geqslant 0$ (non both 0), and $\mathcal{M}\{T\}^\dagger$ is generated by the inverses of the polynomials $1 - \mathrm{L}^a T^b$, for $b > a \geqslant 0$.

An element P of $\mathcal{M}\{T\}^\dagger$ has a *value* $P(\mathrm{L}^{-1})$ at $T = \mathrm{L}^{-1}$.

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Statement of the theorem

1. There exists an integer $a \geqslant 1$, an element $P(T) \in \mathcal{M}\{T\}^{\dagger}$, and an integer $t \geqslant 1$, such that

$$(1 - \mathrm{L}^{a}T^{a})^{t}Z(T) = P(T)$$

   and such that $P(\mathrm{L}^{-1})$ is effective and non-zero.
2. In particular $Z(T)$ belongs to $\mathcal{M}\{T\}$.
3. For every integer $p \in \{0, \ldots, a-1\}$, one of the following cases occurs when d tends to infinity in the congruence class of p modulo a:
   1. Either $\dim(M_d) = \mathrm{o}(d)$,
   2. Or $\dim(M_d) - d$ has a finite limit and $\log(\kappa(M_d))/\log(d)$ converges to some integer in $\{0, \ldots, t-1\}$.

   Moreover, the second case happens at least for one integer p.

Recall that $\kappa(M_d)$ is the number of irreducible components of M_d of maximal dimension.

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The classical Poisson summation formulas

Poisson summation formula: For every smooth rapidly decreasing function f on $\mathbf{R}$, with Fourier transform f, one has

$$\sum_{a\in\mathbf{Z}} f(a) = \sum_{b\in\mathbf{Z}} \widehat{f}(b).$$

More general formula: Let G be a locally compact abelian group, G^* its Pontryagin dual, Γ a discrete cocompact subgroup of G. For matching Haar measures dg on G and $d\chi$ on G^*,

$$\sum_{\gamma\in\Gamma} f(\gamma) = \operatorname{vol}(G/\Gamma) \sum_{\chi\in(G/\Gamma)^*} \widehat{f}(\chi),$$

for every suitable function f on G.

Here, $\hat{f}(\chi) = \int_G f(g)\chi(g)\,dg$ is the Fourier transform of f.

Example: Let C be a connected projective smooth curve over a finite field k, let g be its genus, let $F = k(C)$.
Then F is a discrete cocompact subgroup of the adele group A_F.

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Hrushovski-Kazhdan's motivic Poisson formula

Let C be a connected projective smooth curve over an algebraically closed field k, let g be its genus and $F = k(C)$.

The motivic Poisson formula of Hruhovski-Kazhdan states formally

$$\sum_{x\in F^n} \varphi(x) = \mathbb{L}^{(1-g)n} \sum_{y\in F^n} \mathcal{F}\varphi(y),$$

in which

- $\varphi \in \mathcal{S}$, the space of motivic Schwartz–Bruhat functions on $\mathbb{A}^n_F$, built from suitable relative Grothendieck ring of varieties;
- $\mathcal{F}\varphi \in \mathcal{S}$ is the Fourier transform of φ;
- For $\varphi \in \mathcal{S}$, the sum $\sum_{x\in F^n} \varphi(x)$ is an element of a Grothendieck ring of varieties.

Contents

Varieties with exponentials

Let k be a field. As a group, the Grothendieck ring of k-varieties with exponentials $\mathrm{KExpVar}_k$ is defined by its *generators*:

isomorphism classes pairs $[X, f]$, where X is a k-scheme of finite type and $f: X \to \mathbf{A}^1$ is a morphism,

subject to the *scissor relations*:

$[X, f] = [Y, f|_Y] + [U, f|_U]$, whenever Y is a closed subscheme of X and $U = X \setminus Y$

and to the additional relation:

$[X \times \mathbf{A}^1, \mathrm{pr}_2] = 0$, where pr_2 is the second projection.

The ring structure is defined by

$[X, f] \cdot [Y, g] = [X \times_k Y, \mathrm{pr}_1^* f + \mathrm{pr}_2^* g]$.

Unit element: class $1 = [\mathrm{Spec}(k), 0]$.
Lefschetz element: class L of $[\mathbf{A}^1, 0]$.
Localizations: $\mathcal{M}_k$, $\mathcal{E}xp\mathcal{M}_k$.
The canonical ring morphism $\mathcal{M}_k \to \mathcal{E}xp\mathcal{M}_k$, $[X] \mapsto [X, 0]$ is injective.

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Relative Grothendieck rings with exponentials

Let S be a noetherian k-scheme. One defines analogously a Grothendieck ring $\mathrm{KExpVar}_S$ of S-varieties with exponentials, generated by pairs $[X, f]_S$ where X is an S-scheme of finite type and $f: X \to \mathbf{A}^1$ is a morphism.

A morphism $u: S \to T$ gives rise to:

- A ring morphism $u^*: \mathrm{KExpVar}_T \to \mathrm{KExpVar}_S$, $[X, f]_T \mapsto [X \times_T S, f \circ \mathrm{pr}_1]_S$;
- A group morphism $u_!: \mathrm{KExpVar}_S \to \mathrm{KExpVar}_T$, $[X, f]_S \to [X, f]_T$.

Functional interpretation: An element $\varphi \in \mathrm{KExpVar}_S$ is thought of as a "motivic function" on S, $x \mapsto \varphi(x) = x^*\varphi \in \mathrm{KExpVar}_{k(x)}$.

The morphism u^* corresponds to the composition of functions.

The morphism $u_!$ corresponds to summation over the fibers.

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Local Schwartz–Bruhat functions of given level

Let $F = k(\!(t)\!)$ and let $F^\circ = k[\![t]\!]$. For every integers $N \geqslant M$, $t^M F^\circ / t^N F^\circ$ is identified with the k-rational points of $\mathbf{A}_k^{N-M}$ by the formula

$$x = \sum x_i t^i \pmod{t^N} \mapsto (x_M, \dots, x_{N-1}).$$

Define $\mathcal{S}(F, (M, N)) = \mathcal{E}xp\mathcal{M}_{\mathbf{A}_k^{N-M}}$, the ring of Schwartz–Bruhat functions of level (M, N) on F.

For $\varphi \in \mathcal{S}(F, (M, N))$, one defines

$$\int_F \varphi = \mathrm{L}^{-N} \pi_! \varphi \in \mathcal{E}xp\mathcal{M}_k,$$

where $\pi \colon \mathbf{A}_k^{N-M} \to \operatorname{Spec}(k)$ is the canonical morphism.

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Compatibilities

The closed immersion $\iota\colon \mathbf{A}_k^{N-M} \to \mathbf{A}_k^{N-(M-1)}$, $(x_M, \ldots, x_{N-1}) \mapsto (0, x_M, \ldots, x_{N-1})$ gives rise to two maps:

- restriction: $\iota^*\colon \mathcal{S}(F, (M-1, N)) \to \mathcal{S}(F, (M, N))$;
- extension by zero: $\iota_!\colon \mathcal{S}(F, (N, M)) \to \mathcal{S}(F, (M-1, N))$.

One has $\iota^*\iota_! = \mathrm{I}d$.

The projection $\pi\colon \mathbf{A}_k^{(N+1)-M} \to \mathbf{A}_k^{N-M}$, $(x_M, \ldots, x_N) \mapsto (x_M, \ldots, x_{N-1})$ gives rise to two maps:

- $\pi^*\colon \mathcal{S}(F, (M, N)) \to \mathcal{S}(F, (M, N+1))$;
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Writing

$$F = \varinjlim_{M} \varprojlim_{N \geqslant M} t^M F^\circ / t^N F^\circ,$$

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$$\mathcal{D}(F) = \varprojlim_{M, \iota^*} \varinjlim_{N, \pi^*} \mathcal{S}(F, (M, N)),$$

and its ideal of Schwartz–Bruhat functions:

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“Unit”: $\mathbf{1}_{F^\circ} \in \mathcal{S}(F)$ is induced by the unit element of $\mathcal{S}(F, (0, N))$.

Every $\varphi \in \mathcal{S}(F)$ has an integral $\int_F \varphi \in \mathcal{E}xp\mathcal{M}_k$.
For example, $\int_F \mathbf{1}_{F^\circ} = 1$.

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Fourier transform of local Schwartz–Bruhat functions

Fix $\omega \in \Omega^1_F$, non-zero.

Fourier kernel: $\mathrm{e}(xy) = \mathrm{res}(xy\omega)$. It is defined as an element of $\mathcal{D}(F^2)$ defined by the class $[\mathbf{A}_k^{N-M} \times \mathbf{A}_k^{N'-M'}, u]$, where u is the composition (for suitable N'', M'')

$$\mathbf{A}_k^{N-M} \times \mathbf{A}_k^{N'-M'} \to \mathbf{A}_k^{N''-M''} \to \mathbf{A}_k^1,$$

the first map being induced by the "product" at finite levels, and the second by the residue.

Fourier transform of $\varphi \in \mathcal{S}(F)$: $\quad \mathcal{F}\varphi(y) = \int_F \varphi(x)\mathrm{e}(xy)\,dx.$

It is an element of $\mathcal{S}(F)$.

Fourier inversion: Let ν be the order of the pole of ω. Then

$$\mathcal{F}\mathcal{F}\varphi(x) = \mathrm{L}^{-\nu}\varphi(-x).$$

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Global Schwartz–Bruhat functions

Let C be a connected projective smooth curve over an algebraically closed field k. Let $F = k(C)$; fix a non-zero form $\omega \in \Omega^1_{F/k}$.
For $v \in C(k)$, let $F_v \simeq k(\!(t)\!)$ be the completion of F at v.
The preceding constructions generalize naturally and furnish to finite products of fields isomorphic to $k(\!(t)\!)$.

In particular: every finite set S of $C(k)$, one has a space $\mathcal{S}(\prod_{v\in S} F_v)$, a Fourier transform, a Fourier inversion formula, etc.
For $S' \supset S$, multiplications by "units" $\mathbf{1}_{F_v^\circ}$ for $v \in S' \setminus S$ gives rise to maps $\mathcal{S}(\prod_{v\in S} F_v) \to \mathcal{S}(\prod_{v\in S'} F_v)$.

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Let C be a connected projective smooth curve over an algebraically closed field k. Let $F = k(C)$; fix a non-zero form $\omega \in \Omega^1_{F/k}$.
For $v \in C(k)$, let $F_v \simeq k(\!(t)\!)$ be the completion of F at v.
The preceding constructions generalize naturally and furnish to finite products of fields isomorphic to $k(\!(t)\!)$.

In particular: every finite set S of $C(k)$, one has a space $\mathcal{S}(\prod_{v\in S} F_v)$, a Fourier transform, a Fourier inversion formula, etc.
For $S' \supset S$, multiplications by "units" $\mathbf{1}_{F_v^\circ}$ for $v \in S' \setminus S$ gives rise to maps $\mathcal{S}(\prod_{v\in S} F_v) \to \mathcal{S}(\prod_{v\in S'} F_v)$.

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Summation over rational points

Let $\varphi \in \mathcal{S}(\mathrm{A}_F)$. One wants to define

$$\sum_{x\in F} \varphi(x) \in \mathcal{E}xp\mathcal{M}_k.$$

Assume that φ is represented by an element in $\mathcal{S}(\prod_{v\in S}\mathbf{A}^{N_v-M_v})$.
For an effective divisor D on C, let $\mathcal{L}(D)$ be the corresponding Riemann-Roch space $\mathcal{L}(D)$.

OBSERVATION: F is the inductive limit of the spaces $\mathcal{L}(D)$.

We view $\mathcal{L}(D)$ as a k-scheme and let π_D be the canonical projection to $\mathrm{Spec}(k)$.

For finite sets $S \subset C(k)$, the natural map $\mathcal{L}(D) \to \prod_{v\in S} F_v$ gives rise to a morphisms of schemes

$$u_{D,S} \colon \mathcal{L}(D) \to \prod_{v\in S} \mathbf{A}_k^{N_v-M_v}.$$

For large enough D, one can then set

$$\sum_{x\in F} \varphi(x) = (\pi_D)_!(u_{D,S})^*\varphi \in \mathcal{E}xp\mathcal{M}_k.$$

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We now can state Hrushovski–Kazhdan's motivic Poisson formula: For every $\varphi \in \mathcal{S}(\mathrm{A}_F)$, one has

$$\sum_{x\in F} \varphi(x) = \mathrm{L}^{1-g} \sum_{y\in F} \mathcal{F}\varphi(y).$$

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1. One extends the height, $G(F) \to \mathbf{R}$, to a function on the adelic space $G(\mathrm{A}_F) \to \mathbf{R}$.
2. Recall that $G(F)$ is a cocompact discrete subgroup of $G(\mathrm{A}_F)$. Fixing a non-trivial additive character ψ of A_F, then $(G(\mathrm{A}_F)/G(F))^*$ is identified with $G(F)$ by the pairing $\langle x, y\rangle = \psi(xy)$.

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$$Z(s) = \sum_{x \in G(F)} H(x)^{-s} = \sum_{y \in G(F)} \mathcal{F}(H(\cdot)^{-s})(y).$$

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6. This gives meromorphic continuation for $Z(s)$. A tauberian theorem allows to conclude.

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Extending the height function

The first step consists in extending the height function to $G(\mathbb{A}_F)$.
We are given a model $\mathcal{X} \to C$ of X. Moreover, L is linearly equivalent to a divisor which does not meet G. Consequently, $\mathcal{L}$ is linearly equivalent to a divisor Δ on $\mathcal{X}$ which does not meet G.

$$h(x) = \sum_{v \in C} (\Delta, \sigma_x(C))_v,$$

a sum of local intersection numbers.

Since Schwartz–Bruhat functions need to be $1_{F_v^\circ}$ at almost all places, we are "forced" to consider an analogous problem of counting points such that moreover $(\Delta, \sigma_x(C))_v = 0$ for all $v \in C_0$ — we do this by imposing that $\sigma_x(C_0) \subset \mathcal{U}$.

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Applying the Poisson summation formula

One finds a finite subset $S \supset C(k)$, and for $v \in S$, motivic Schwartz–Bruhat functions $(\varphi_{v,m})_{m \in \mathbf{Z}}$ with disjoint supports and zero for $m \ll 0$ such that for each integer d, the condition

“$h(x) = d$ and $\sigma_x(C_0) \subset \mathcal{U}$”

are equivalent to the condition

“$\varphi_{v,m_v}(x) = 1$ for some $\mathbf{m} = (m_v) \in \mathbf{Z}^S$ with $|\mathbf{m}| = \sum m_v = d$”

Then

$$Z(T) = \sum_{x \in G(F)} \prod_{v \in S} \left(\sum_{m \in \mathbf{Z}} \varphi_{v,m}(x) T^m \right) = \sum_{\mathbf{m}} \sum_{x \in G(F)} \varphi_{\mathbf{m}}(x) T^{|\mathbf{m}|}.$$

Applied to each $\varphi_{\mathbf{m}}$, Hrushovski–Kazhdan's formula gives

$$Z(T) = \mathrm{L}^{(1-g)n} \sum_{y \in G(F)} \hat{Z}(T; y),$$

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For $y = 0$, each $\hat{Z}_v(T; 0)$ is a kind of motivic Igusa zeta function. Its analysis is classical in motivic integration and furnishes rational functions.

For general y, we obtain a motivic analogue of the oscillatory integrals

$$\int |f(x)|^s \exp(2i\pi g(x))\, dx.$$

One obtains rational functions again, with smaller poles than for $y = 0$. The vanishing of the motivic integral

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In the formula

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one shows:

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