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17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: James Freitag **Email/Phone:** freitag@math.berkeley.edu

Speaker's Name: Diedre Haskell

Talk Title: Towards a theory of residue field domination for convexly valued ordered fields

Date: 05/12/14 **Time:** 11:00 am / pm (circle one)

List 6-12 key words for the talk: Valued Fields, ordered fields, convex subrings, o-minimal, T-convex expansions, model theory

Please summarize the lecture in 5 or fewer sentences: This is a report on recent developments on to what extent a T-convex expansion of a real closed valued field is determined by its value group and residue field.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
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Towards a theory of residue field domination for convexly valued ordered fields

Deirdre Haskell
presenting joint work with Clifton Ealy and Jana Maříková
supported by NSERC and MSRI

McMaster University

MSRI program on model theory, number theory and arithmetic geometry
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convex subring of an ordered field

Let $\mathcal{R}$ be an ordered field, V a convex subring; that is, for all $x, y \in \mathcal{R}$,

$$\text{if } x, y \in V \text{ and } x < z < y \text{ then } z \in V.$$

Then V is a valuation ring: for let $x \in \mathcal{R}$ and assume that $x > 0$.
If $0 < x < 1$ then $x \in V$ by convexity.
If $1 < x$ then $0 < 1/x < 1$, so $1/x \in V$.
Thus $(\mathcal{R}, V)$ is a valued field.
Write Γ for the value group, $v : \mathcal{R}^\times \to \Gamma$,
k for the residue field, $\mathrm{res} : V \to k$.

The smallest convex subring is the convex hull of $\mathbb{Z}$: $\mathcal{O}_\mathbb{Z}$.
If $\mathcal{O}_\mathbb{Z} \neq \mathcal{R}$ then $\mathcal{R}$ is non-archimedean.
Take $x \in \mathcal{R} \setminus \mathcal{O}_\mathbb{Z}$: then $v(x) < 0$, so $v(1/x) > 0$.
Thus $\mathfrak{m}_\mathbb{Z}$ is a convex set of infinitesimals around 0.

convex subring of an ordered field

Picture:

Around every $a \in \mathcal{O}_{\mathbb{Z}}$ is a convex set $\{x : v(x - a) > 0\}$, which is a cone around a.

The tree of the valued field arises naturally, with the ordering on $\mathcal{R}$ inducing the ordering on k — or vice versa.

closure properties

1) If $\mathcal{R}$ is real closed then Γ is divisible and k is real closed.

2) If Γ is divisible, k is real closed and $\mathcal{R}$ is henselian then $\mathcal{R}$ is real closed.

Analogous to algebraically closed valued fields.

model theory of pure real closed valued fields

Cherlin-Dickmann 1983

The theory RCVF of real closed fields with a proper convex valuation has quantifier elimination in the language of rings with predicates for the ordering and the valuation.

Mellor 2006

The theory RCVF has elimination of imaginaries in the sorted language $\mathcal{G}$ with sorts for the finitely generated $\mathcal{O}_{\mathcal{R}}$-submodules and their torsors.

Again, analogous to algebraically closed valued fields.

T-convex theories

Definition

Let T be a complete o-minimal theory in a language extending $\{+, -, \cdot, 0, 1, <\}$. Let $\mathcal{R} \models T$, V be a convex subring of $\mathcal{R}$. Say that V is *T-convex* if for every $\emptyset$-definable continuous function $f : \mathcal{R} \to \mathcal{R}, f(V) \subseteq V$.

Theorem (van den Dries-Lewenberg 1995)

Let $\mathcal{R} \models T$, where T is a complete o-minimal theory with quantifier elimination. Let $V \subset \mathcal{R}$ be a proper T-convex subring. Then $Th(\mathcal{R}, V)$ also has quantifier elimination.

power bounded o-minimal theories

Let $\mathcal{R} \models T$, T a complete o-minimal theory

Definition

A *power function* on $\mathcal{R}$ is a function f such that $f(xy) = f(x)f(y)$ for all positive x, y. We can think of a power function as $x \mapsto x^r$, for some $r \in \mathcal{R}$. The *field of exponents* of $\mathcal{R}$ is the set of $f'(1) \in \mathcal{R}$ such that f is a definable power function in $\mathcal{R}$.

$\mathcal{R}$ is *power bounded* if for every definable function $f : \mathcal{R} \to \mathcal{R}$ there is r in the field of exponents of $\mathcal{R}$ such that, for all sufficiently large x, $f(x) < x^r$.

consequences of power bounded

From now on, assume that T is a power bounded o-minimal theory with field of exponents F, $\mathcal{R} \models T$.

van den Dries 1997

1) The residue field k, with the structure induced from $\mathcal{R}$, is a model of T.
k is stably embedded in $\mathcal{R}$: if $S \subset V^n$ is definable then $\mathrm{res}(S) \subseteq k^n$ is definable in k.

2) The theory of Γ, with structure induced from $\mathcal{R}$, is the theory of non-trivial, ordered vector spaces over F, so in particular is o-minimal.
Γ is stably embedded in $\mathcal{R}$: if $S \subset \mathcal{R}^{\times n}$ is definable then $v(S) \subseteq \Gamma^n$ is definable in Γ.

3) (Wilkie Inequality) Suppose $\mathcal{R} \preceq \mathcal{R}'$. Then
$\mathrm{rk}(\mathcal{R}'/\mathcal{R}) \geq \mathrm{rk}(k(\mathcal{R}')/k(\mathcal{R})) + \dim_F(\Gamma(\mathcal{R}')/\Gamma(\mathcal{R}))$.

aside: V convex but not necessarily T-convex

Theorem

Let $\mathcal{R}$ be an o-minimal expansion of a real closed field in a language in which $\mathcal{R}$ eliminates quantifiers, V a convex subring.

1) (Maříková 2009, 2010) The induced structure on k is o-minimal if and only if $(\mathcal{R}, V)$ models a given recursive list of axioms.

2) (Ealy-Maříková 2013) Fix $(\mathcal{R}_0, V_0)$ an elementary substructure of $(\mathcal{R}, V)$. Suppose the induced structure on k is o-minimal. Then $(\mathcal{R}, V)$ is model-complete in the language for $\mathcal{R}$ expanded by V and constants from $\mathcal{R}_0$.

stable

In general, $\mathcal{U}$ a saturated model of a complete theory T, $C \subset U$ an algebraically closed set of parameters.

In a stable theory, we have a notion of independence: $A \downarrow_C B$.

If $A \downarrow_C B$ then whenever $A \equiv_C A'$ and $A' \downarrow_C B$ also $A \equiv_B A'$; that is, if there is $\sigma \in \mathrm{Aut}(\mathcal{U}/C)$ with $\sigma(A) = A'$ then there is $\tau \in \mathrm{Aut}(\mathcal{U}/B)$ with $\tau(A) = A'$.

An algebraically closed field is stable.
An ordered field is not stable.
A valued field is not stable.

stable, stably embedded

Definition

1) The C-definable set D is *stable* if the structure with universe D and relations all of the C-definable subsets of D is stable.
Example: the residue field in an algebraically closed valued field

2) The C-definable set D is *stably embedded* if for any definable set E, $E \cap D^n$ is $C \cup D$-definable.
Example: the value group in an algebraically closed valued field

3) The stable part (of $\mathcal{U}$) over C, St_C is the multisorted structure (D_i, R_{ij}) with sorts all the C-definable stable, stably embedded sets and relations all the C-definable relations on D_i.
Example: in an algebraically closed valued field, the stable, stably embedded sets are those which are defined, with only finitely many extra parameters, over the residue field.

stable domination

Definition

$\mathrm{tp}(A/C)$ is *stably dominated* if, whenever $\mathrm{St}_C(A) \downarrow_{\mathrm{St}_C} \mathrm{St}_C(B)$ then

$$\mathrm{tp}(A/C\mathrm{St}_C(B)) \vdash \mathrm{tp}(A/CB).$$

That is, if there is $\sigma \in \mathrm{Aut}(\mathcal{U}/C\mathrm{St}_C(B))$ with $\sigma(A) = A'$ and $\mathrm{St}_C(A') \downarrow_{\mathrm{St}_C} \mathrm{St}_C(B)$ then there is $\tau \in \mathrm{Aut}(\mathcal{U}/CB)$ with $\tau(A) = A'$.

stable domination in ACVF

Theorem (H.-Hrushovski-Macpherson 2008)

Let $\mathcal{U}$ be a saturated model of the theory of algebraically closed valued fields. Let C be a maximally complete submodel. Let a be a tuple from U and $A = \mathrm{acl}(Ca)$. Then $\mathrm{tp}(A/C\Gamma(A))$ is stably dominated.

That is, if $\mathrm{St}_{C\Gamma(A)}(A) \downarrow_{\mathrm{St}_{C\Gamma(A)}} \mathrm{St}_{C\Gamma(A)}(B)$ and there is $\sigma \in \mathrm{Aut}(\mathcal{U}/C\Gamma(A)\mathrm{St}_{C\Gamma(A)}(B))$ with $\sigma(A) = A'$ then there is $\tau \in \mathrm{Aut}(\mathcal{U}/C\Gamma(A)B)$ with $\tau(A) = A'$.

Theorem (Hrushovski, unpublished notes about 2010)

Let $\mathcal{U}$ be a saturated model of a C-minimal theory in a language expanding the language of valued fields. Then the same holds.

residue field domination

In a model $\mathcal{R}$ of a T-convex theory: replace $\mathrm{St}_C(A)$ by $k(A)$, replace $\downarrow$ by o-minimal independence.

Definition

$\mathrm{tp}(A/C)$ is *dominated by its residue field* if, whenever $k(A) \downarrow_{k(C)} k(B)$ then

$$\mathrm{tp}(A/Ck(B)) \vdash \mathrm{tp}(A/CB).$$

a theorem

Theorem (Ealy-Haskell-Maříková 2014)

Let L, M, C be T-convex structures, where T is a power bounded o-minimal theory. Assume that C is maximally complete, a substructure of both L and M and the $\mathrm{trdeg}(L/C)$ is finite. Assume that $k(L) \downarrow_{k(C)} k(M)$ and $\Gamma(L) \downarrow_{\Gamma(C)} \Gamma(M)$. Let $\sigma : L \to L'$ be an isomorphism fixing $Ck(L)\Gamma(L)$. Assume also that $k(L') \downarrow_{k(C)} k(M)$. Then σ extends by the identity on M to an isomorphism from $\langle L, M\rangle$ to $\langle L', M\rangle$, where $\langle L, M\rangle$ is the o-minimal structure generated by L and M.

That is, $\mathrm{tp}(L/C)$ is dominated by its residue field and value group.

(Here, independence in the o-minimal theory is in the sense of thorn-independence; that is, the algebraic rank of a tuple does not decrease over the larger set of parameters.)

outline of proof

By induction on $n = \text{trdeg}(L/C)$.
For the $n+1$ step, write $L_n = \langle C, \ell_1, \ldots, \ell_n \rangle$, $A_n = \langle M, \ell_1, \ldots, \ell_n \rangle$ and assume the isomorphism $\sigma : L_n \to L'_n$ extends to an isomorphism from A_n to A'_n. It follows from the hypotheses on $k(L)$ and $\Gamma(L)$ that $\Gamma(A_n) = \Gamma((L_n, M))$.

1) Extend the isomorphism $\sigma : L_n \to L'_n$ to a valued field isomorphism from (M, L_n, ℓ) to (M, L'_n, ℓ'): this is a small modification to proof of [HHM].

2) Show that the isomorphism also preserves the ordering:
over a maximally complete base C, can find a nearest element; this extends to a finitely generated vector space over C.

outline of proof

3) Now extend σ to (A_n, ℓ):
if ℓ adds a new element to either the value group or the residue field (but not both), then can use the same arguments as for steps 1) and 2). Otherwise, by the Wilkie Inequality, (A_n, ℓ) is an immediate extension of A_n. Properties of o-minimal definable functions show that the order type of ℓ over A_n is determined by the order type of ℓ over (M, L_n), and hence 2) gives an order isomorphism of (A_n, ℓ).

4) Extend σ to $\langle A_n, \ell \rangle$:
set $\sigma(f(\bar{a}, \ell)) = f(\sigma(\bar{a}), \sigma(\ell))$ for any $\emptyset$-definable function f, and show this is well-defined using properties of o-minimal definable functions.

5) Conclude that σ preserves the ordering and hence the valuation.

a stronger theorem

Theorem (Ealy-Haskell-Maříková 2014)

Let L, M, C be T-convex structures, where T is a power bounded o-minimal theory. Assume that C is maximally complete, a substructure of both L and M and the $\mathrm{trdeg}(L/C)$ is finite. Assume that $k(L) \downarrow_{k(C)} k(M)$ and $\Gamma(L) \subseteq \Gamma(M)$. Let $\sigma : L \to L'$ be an isomorphism fixing $Ck(L)\Gamma(L)$. Assume also that $k(L') \downarrow_{k(C)} k(M)$. Then σ extends by the identity on M to an isomorphism from $\langle L, M \rangle$ to $\langle L', M \rangle$, where $\langle L, M \rangle$ is the o-minimal structure generated by L and M.

That is, $\mathrm{tp}(L/C\Gamma(L))$ is dominated by its residue field.

As in [HHM], perturb the valuation, to move into the situation of the previous theorem.

further directions

Extend to the geometric sorts, in which RCVF eliminates imaginaries.

What about other imaginaries in a T-convex theory?

What is the structure of the residue field dominated types?