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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: James Freitag **Email/Phone:** freitag@math.berkeley.edu

Speaker's Name: Silvain Rideau

Talk Title: Transferring imaginaries: from ACVF to Qp

Date: 05/12/14 **Time:** 3:30 am / pm (circle one)

List 6-12 key words for the talk: elimination of imaginaries, p-adics valued fields, ACVF, model theory, rational points

Please summarize the lecture in 5 or fewer sentences: This talk explains how to transfer elimination of imaginaries from algebraically closed valued fields to the p-adics. The transfer actually holds in a more general abstract setting. The general result also has applications to elimination of imaginaries in ultraproducts, which yields certain uniformity results

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Transferring imaginaries

How to eliminate imaginaries in p-adic fields

Silvain Rideau

joint work with E. Hrushovski and B. Martin
in “Definable equivalence relations and zeta functions of groups”
with an appendix by R. Cluckers

Orsay Paris-Sud 11, École Normale Supérieure

May 12, 2014

Some notations

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- The value group will be denoted by Γ;
- Let also $\mathrm{RV} := K^\star/(1 + \mathfrak{M}) \supseteq k^\star$.

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Let $\mathcal{L}_{\mathrm{P}} = \mathcal{L}_{\mathrm{div}} \cup \{P_n \mid n \in \mathbb{N}_{>0}\}$ where $x \in P_n$ if and only if $\exists y,\ y^n = x$.

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Theorem (Macintyre, 1976)

The $\mathcal{L}_{\text{P}}$-theory of $\mathbb{Q}_p$ eliminates quantifiers.

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Theorem (Poizat, 1983)

The theory ACF of algebraically closed fields in the language $\mathcal{L}_{\mathrm{rg}} = \{\mathbf{K}; 0, 1, +, -, \cdot\}$ eliminates imaginaries.

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Positive answers to these two questions are equivalent and is called elimination of imaginaries.

Remark

To any $\mathcal{L}$-structure M we can associate the $\mathcal{L}^{\mathrm{eq}}$-structure M^{eq} where we add a point for each imaginary.

Imaginaries in valued fields

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In the language $\mathcal{L}_{\text{div}}$, the quotient $\Gamma = \mathbf{K}^\star / \mathcal{O}^\star$ is not representable in algebraically closed valued field nor in $\mathbb{Q}_p$.

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However, in the case of ACVF — the theory of algebraically closed valued fields — Haskell, Hrushovski and Macpherson have shown what imaginary sorts it suffices to add.

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The geometric language $\mathcal{L}_{\mathcal{G}}$ is composed of the sorts $\mathbf{K}$, $\mathbf{S}_n$ and $\mathbf{T}_n$ for all n, with $\mathcal{L}_{\mathrm{rg}}$ on $\mathbf{K}$ and functions $\rho_n : \mathrm{GL}_n(\mathbf{K}) \to \mathbf{S}_n$ and $\tau_n : \mathbf{S}_n \times \mathbf{K}^n \to \mathbf{T}_n$.

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- $\mathbf{S}_1$ can be identified with Γ and ρ_1 with v;
- $\mathbf{T}_1$ can be identified with RV;
- The set of balls (open and closed, possibly with infinite radius) $\mathbb{B}$ can be identified with a subset of $\mathbf{K} \cup \mathbf{S}_2 \cup \mathbf{T}_2$.

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- The $\mathcal{L}_{\mathcal{G}}$-theory $\mathrm{ACVF}^{\mathcal{G}}$ eliminates imaginaries.
- In particular, the imaginaries in $\mathrm{ACVF}^{\mathcal{G}}_{0,p}$ (respectively those in $\mathrm{ACVF}^{\mathcal{G}}_{p,p}$) can be eliminated uniformly in p.

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Question

1. Are all imaginaries in $\mathbb{Q}_p$ coded in the geometric sorts or are there new imaginaries in this theory?
2. Can these imaginaries be eliminated uniformly in p?

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Similarly for acl, tp and TP (the space of types).

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Remark

Every $\prod K_p/\mathcal{U}$ where K_p is a finite extension of $\mathbb{Q}_p$ and $\mathcal{U}$ is a non principal ultrafilter on the set of primes is a model of PLF. In fact, By the Ax-Kochen-Eršov principle any model of PLF is equivalent to one of these ultraproducts.

A first example: extracting square roots in $\mathbb{Q}_3$

- Let $a \in \mathbb{Q}_3$ and $f : P_2(\mathbb{Q}_3^\star) + a \to \mathbb{Q}_3$, where P_2 is the set of squares, defined by:

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- The graph of f is coded by the code of F.

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Then T eliminates imaginaries.

Another abstract criterion

Theorem

Assume the following holds:

(i) Any $\mathcal{L}(M)$-definable unary set $X \subseteq \mathbf{K}(M)$ is coded;

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(iii) For all $e \in \mathrm{dcl}_{\widetilde{M}}(M)$, there exists a tuple $e' \in M$ such that for all $\sigma \in \mathrm{Aut}(\widetilde{M})$ with $\sigma(M) = M$, σ fixes e if and only if it fixes e';

(iv) For any $A = \mathrm{acl}^{\mathrm{eq}}_M(A) \cap M$ and $c \in \mathbf{K}(M)$, there exists an $\mathrm{Aut}(\widetilde{M}/A)$-invariant type $\widetilde{p} \in \mathrm{TP}_{\widetilde{M}}(\widetilde{M})$ such that $\widetilde{p}|M$ is consistent with $\mathrm{tp}_{\mathcal{L}}(c/A)$;

(v') For all $A \subseteq M$ and any $e \in \mathrm{acl}^{\mathrm{eq}}_M(A)$ there exists $e' \in M$ such that $e \in \mathrm{dcl}^{\mathrm{eq}}_M(Ae')$ and $e' \in \mathrm{dcl}^{\mathrm{eq}}_M(Ae)$.

Then T eliminates imaginaries.

p-adic imaginaries

Theorem

Let K be a finite extension of $\mathbb{Q}_p$, then the theory of K in the language $\mathcal{L}_{\mathcal{G}}$ with a constant added for a generator of $K \cap \overline{\mathbb{Q}}^{\text{alg}}$ over $\mathbb{Q}_p \cap \overline{\mathbb{Q}}^{\text{alg}}$ eliminates imaginaries.

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Proof.

It follows from the first EI criterion. □

Ultraproducts

Theorem

Let $K = \prod K_p/\mathcal{U}$ be an ultraproduct of finite extensions K_p of $\mathbb{Q}_p$. The theory of K in the language $\mathcal{L}_{\mathcal{G}}$, with constants added for a uniformizer and an unramified Galois-uniformizer, eliminate imaginaries.

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Proof.

It follows from the second EI criterion. □

Remark

The sorts T_n are useless in those two cases.

Uniformity

Let $\mathcal{L}_{\mathcal{G}}^{\star}$ be $\mathcal{L}_{\mathcal{G}}$ with two constants in $\mathbf{K}$ added.

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Corollary

For any equivalence relation E_p on a set D_p definable in K_p uniformly in p, there exists m_0 and an $\mathcal{L}_{\mathcal{G}}^{\star}$-formula $\phi(x,y)$ such that for all p, ϕ defines a function

$$f_p : D \to K_p^l \times S_m(K_p)$$

where K_p is made into a $\mathcal{L}_{\mathcal{G}}^{\star}$-structure by choosing a uniformizer and an unramified m_0-Galois uniformizer and

$$K_p \vDash \forall x,y,\ xE_py \iff f_p(x) = f_p(y).$$

Definable families of equivalence relations

Fix p a prime and let K_p be a finite extension of $\mathbb{Q}_p$.

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Definition

A family $(R_l)_{l \in \mathbb{N}^r} \subseteq K_p^n$ is said to be uniformly definable if there is an $\mathcal{L}_{\mathcal{G}}$ formula $\phi(x, y)$ such that for all $l \in \mathbb{N}^r$,

$$\phi(K_p, l) = R_l.$$

We say that $E \subseteq R^2$ is a definable family of equivalence relations on R if E is an equivalence relation on R and

$$\forall x, y \in R,\ xEy \Rightarrow \exists l \in \mathbb{N}^r,\ x, y \in R_l.$$

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In particular, for all $l \in \mathbb{N}^r$, E induces an equivalence relation E_l on R_l.

Definable families of equivalence relations

For all prime p, let K_p be a finite extension of $\mathbb{Q}_p$.

Definition

A family $(R_{p,l})_{l\in\mathbb{N}^r} \subseteq K_p^n$ is said to be definable uniformly in p if there is an $\mathcal{L}_{\mathcal{G}}$ formula $\phi(x,y)$ such that for all prime p and $l \in \mathbb{N}^r$,

$$\phi(K_p, l) = R_{p,l}.$$

We say that $E_p \subseteq R_p^2$ is a family of equivalence relations on R_p definable uniformly in p if E_p is an equivalence relation on R_p and

$$\forall p \forall x, y \in R_p,\ xE_py \Rightarrow \exists l \in \mathbb{N}^r,\ x, y \in R_{p,l}.$$

In particular, for all $l \in \mathbb{N}^r$, E_p induces an equivalence relation $E_{p,l}$ on $R_{p,l}$.

Rationality

Theorem

Fix p a prime. Let $(R_\nu)_{\nu\in\mathbb{N}^r} \subseteq K_p^n$ be uniformly definable and E a family of definable equivalence relations on R such that for all $l \in \mathbb{N}^r$, $a_\nu = |R_\nu/E_\nu|$ is finite. Then

$$\sum_\nu a_\nu t^\nu \text{ is rational.}$$

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$$\sum_\nu a_{p,\nu} t^\nu \text{ is rational.}$$

Moreover, there exists m_0 and $d \in \mathbb{N}$ such that for all choice of m_0-Galois uniformizer $c_p \in K_p$, for all $\nu \in \mathbb{N}^r$ with $|\nu| \leq d$, there exists $q_\nu \in \mathbb{Q}$ and varieties V_ν and W_ν over $\mathbb{Z}[X]$ such that for all $p \gg 0$,

$$\sum_\nu a_{p,\nu} t^\nu = \frac{\sum_{|\nu|\leq d} q_\nu |V_\nu(\mathrm{res}(K_p))| t^\nu}{\sum_{|\nu|\leq d} |W_\nu(\mathrm{res}(K_p))| t^\nu}$$

where X is specialized to $\mathrm{res}(c_p)$ in $\mathrm{res}(K_p)$.

Some remarks

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- In the appendix, Raf Cluckers gives an alternative proof of the counting theorem for fixed p that does not use elimination of imaginaries and generalizes to the analytic setting.
- The denominator of the rational function can described more precisely.
- These results are used to show that some zeta functions that appear in the theory of subgroup growth and representation growth are rational uniformly in p.

Thank you