Lehmer Problem and Applications
Presentation by Maria Carrizosa

1 Zilber-Pink Conjecture

The Lehmer problem is one of the "special points/ special varieties" problems.

We want to find all rational points of a polynomial of the form $p(x, y) = 0$ which gives us a smooth projective curve $X/\mathbb{Q}$. We know that the number of $\mathbb{Q}$ points of these solutions is determined by its genus g.

1. $g = 0$. Then $X(\mathbb{Q})$ is either empty or infinite, and its solutions are rationally parametrized.
2. $g = 1$. Then $X(\mathbb{Q})$ is an elliptic curve, and the Mordell-Weil theorem asserts that $X(\mathbb{Q})$ is a finitely generated group. More generally, for A an abelian variety, $A(\mathbb{Q})$ is finitely generated.
3. $g > 1$, $X(\mathbb{Q})$ is finite. For every such curve, there is an embedding of X into its *Jacobian* $J(X) := A$. an abelian variety of dimension g and consider $X \cap A(\mathbb{Q})$ in this embedding. This situation is what the Mordell-Lang conjecture generalizes.

This leads to the statement of the **Mordell-Lang Conjecture**: Let X be a curve contained in an abelian variety A, Γ a finite rank subgroup (i.e. $\Gamma \otimes \mathbb{Q}$ is a f.d. $\mathbb{Q}$ vector space). Then $X \cap \Gamma$ is finite except if X is a translate of an elliptic curve.

Zilber-Pink Conjecture Let A be an (semi-)abelian variety over K a number field, Γ a finite-rank subgroup of $A(\overline{K})$, X a subvariety that is Γ transverse (i.e. X is *not* contained in a translate by points of $\Gamma_{sat} = \{x \in A \,|\, \exists n \in \mathbb{Z} n \cdot x \in \Gamma\}$ of a proper abelian subvariety of A). Consider the sets of the form $X \cap (\Gamma + B)$ (with codim$B \geq \dim X + 1$); none of these are Zariski dense in X by Mordell-Lang. Then the conjecture says that $X \cap (\Gamma + \bigcap_{\text{abelian subgroups of } A \text{ such that codim} B \geq \dim X+1} B)$ is not dense in X

Equivalently, we can suppose that $\Gamma = \{0\}$, *or* that X is A-transverse.

The main first steps in this direction are

Theorem 1. *(Bombieri-Masser-Zannier and Manin) The Zilber-Pink conjecture is true for* $A = \mathbb{G}_m^n$, $X = C$ *a curve.*

Theorem 2. *(Habegger-Pila)* A *an abelian variety,* $X = C$ *a curve.*

How can we try to prove this conjecture?

2 A Strategy

Very general idea: We're studying the intersection of X with some set S; $X \cap S$. We try to...

1. Find an "exceptional" $Z \subseteq X$ closed such that
2. Show that $(X \setminus Z) \cap S$ has bounded height

3. Show that $(X \setminus Z) \cap S$ is finite

4. Show that, in fact, $X = Z$.

Definition 1. For any $\Gamma \subseteq A(\overline{K})$, let

$Z_{X,\Gamma} = \{x \in X \mid \exists H$ a $\Gamma-$torsion subvariety such that $\dim_x(X \cap H) > \max(0, \dim X + \dim H - \dim A)\}$

People who worked on each step...

1. Step one done for $\mathbb{G}_m^n$ by Bombieri-Masser-Zannier for $\Gamma = A$ for A an abelian subgroup. Also done for A an abelian variety by G. Remond.

2. Step two done for $\mathbb{G}_m^n$ by Habegger for $\Gamma = 0$ and by Manin for general Γ. Also done for abelian varieties A by Habegger for $\Gamma = 0$ and also by Remond.

3. Step 3- Lehmer Problem

4. For step four, all we really know is that this step works for curves.

3 Heights and Dobrowolski group

We begin by defining heights. Let $\frac{a}{b} \in \mathbb{Q}$ such that $(a, b) = 1$. Then defin $h(\frac{a}{b}) = \log \max(|a|, |b|)$. Now let $\alpha \in \overline{\mathbb{Q}}$. Let $\mu_\alpha =$ minimal polynomial of αover $\mathbb{Z} = \sum a_i x^i$. Define the *naive* height of α as

$$h_{naive}(\alpha) = \frac{1}{d} \log \max |a_i|$$

and then define the height as $h(\alpha) = \lim_{n\to\infty} \frac{1}{n} h_{naive}(\alpha^n)$. We also have that

$$h(\alpha) = \sum_{\nu \in \mathfrak{M}_K} \frac{[K_\nu : \mathbb{Q}_\nu]}{[K : \mathbb{Q}]} \log \max(1, |\alpha|_\nu)$$

, $\alpha \in \overline{\mathbb{Q}}^*$.

Some easy facts about height:

- $h(\alpha) = 0$ if and only if α is a root of unity
- $h(\alpha^n) = |n| h(\alpha)$ for $n \in \mathbb{Z}$.

Lehmer's Conjecture (1933) $\exists c > 0$, $\alpha \in \overline{\mathbb{Q}}^*$, α not a root of unity such that $h(\alpha) \geq \frac{c}{[\mathbb{Q}(\alpha) : \mathbb{Q}]}$

Theorem 3. *(Dobrowolski 1979)* $\forall \epsilon > 0$ $\exists c(\epsilon) > 0$, $\alpha \in \overline{\mathbb{Q}}^*$, $\alpha \in \mu$ *with* $D^{1+\epsilon} h(\alpha) \geq c$ *where* $D = [\mathbb{Q}(\alpha) : \mathbb{Q}]$.

4 Lehmer and Dobrowolski Groups

First we define the **obstruction index**, a generalization of heights: Consider $K \subseteq L \subseteq \overline{K}$, $x \in A$. Define the obstruction index of L over K as

$$w_L(x) = \min\{\deg \mathcal{L}(V)^{1/[\dim A - \dim V]}, x \in V/L \subseteq A\}$$

taking $L = \mathbb{Q}$ and $V = \{x\}$, the degree $\deg_{\mathcal{L}}$ is simply $[\mathbb{Q}(x) : \mathbb{Q}]$ and we recover height.

Remark: The line bundle $\mathcal{L}$ above comes from a choice of embedding our variety A in projective space.

Let $K_\Gamma =$ field of rationality of Γ.

Definition 2. Γ is a Lehmer group if $\exists c(\Gamma) > 0$ $\forall x$ that are Γ-transverse we have $w_{K_\Gamma}(x)h(x) \geq c_\Gamma$.

Examples:

- Take $\Gamma = \{0\}$, $K_\Gamma = K$, $\Gamma_{sat} = A_{tors}$. This is called the **Lehmer problem**).
- Likewise, let $\Gamma = A_{tors}$, $K_\Gamma = K(A_{tors})$. This is the *relative* Lehmer problem.
- Take $\Gamma = A$, so $K_\Gamma = \overline{K}$, which is the effective Bogomolov conjecture.

What we know:

Definition 3. Γ is a Dobrowolski group if $\exists \epsilon > 0$, $\exists c_\epsilon(\Gamma) > 0$ $\forall x$ that are Γ-transverse we have $w_{K_\Gamma}(x)h(x) \geq c_{\epsilon,\Gamma}$.

Theorem 4. *Let $A = \mathbb{G}_m^n$, A an abelian variety with CM. Then $\Gamma = \{0\}, A_{tors}, A$ are Dobrowolski.*

5 Back to Zilber-Pink Conjecture

Theorem 5. *If A_{tors} is Dobrowolski, A is Dobrowolski, and $X \neq Z_{x,A}$ then the Zilber-Pink conjecture in that case is true.*

If $\alpha \in \mathbb{Q}^{ab}$ then $h(\alpha) \geq \dfrac{\log 5}{12}$ so that the Lehmer problem is true for such α. Another sufficient condition, which comes by easy bounds on height, is that is α^{-1} is not a conjugate of α.