Counting Algebraic Points on Definable Sets
Presentation by Margaret Thomas

1 Some Definitions

Definition 1 (Heights)**.** For $\dfrac{a}{b} \in \mathbb{Q}$ such that $(a,b)=1$, $h(\dfrac{a}{b}) = \log\max(|a|,|b|) := \log H$.

Definition 2 (Pfaffian)**.** Consider a sequence $f_1, \cdots, f_r : U \to \mathbb{R}$ of analytic functions ($U \subseteq \mathbb{R}^n$ open) such that $\dfrac{\partial f_i}{\partial x_j}(\overline{x}) = P_{ij}(\overline{x}, f_1(\overline{x}), \cdots, f_i(\overline{x}))$.

A function $f(\overline{x})$ is Pfaffian if we can write it as the solution to a polynomial differential equation $f(\overline{x}) = P(\overline{x}, f_1(\overline{x}), \cdots, f_r(\overline{x}))$ with $\deg P_{ij} \leq \alpha$, $\deg P \leq \beta$.

2 Strategy For Pila-Wilkie over Number Fields

(1) Parametrization: Cover $S \subseteq \mathbb{R}^n$ by $\bigcup_{i=1}^{n} Im(\phi_i)$ for $\phi_i : (0,1)^{\dim S} \to \mathbb{R}^n$ with ϕ_i chosen so that we have nice bounds on the absolute values of the derivatives of the ϕ_i's.

(2) Diophantine Part: Reduce from considering the set of points $S(F,H) := \{\overline{q} \in S \cap F^n \,|\, ht(\overline{q}) \leq H\}$ to counting $(S \cap Z(P))(F,H)$ for polynomials of suitable degree, F a number field.

(3) Zero Estimates: Count $|S \cap Z(P)|$ or $|(S \cap Z(P))(F,H)|$.

Counting algebraic points on definable sets

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Pila–Wilkie Theorem

Let $S \subseteq \mathbb{R}^n$ be definable in an o-minimal expansion of the ordered field of real numbers. Assume that S contains no infinite semialgebraic subset. Let $\varepsilon > 0$. There exists $C = C(\varepsilon) > 0$ such that if $H \geq C$, then S contains at most H^ε rational points of height at most H, i.e. setting $S(\mathbb{Q}, H) := \{\bar{q} \in S \cap \mathbb{Q}^n \mid \operatorname{ht}(\bar{q}) \leq H\}$, we have that for $H \geq C$, $|S(\mathbb{Q}, H)| \leq H^\varepsilon$.

(Pila 2009) The theorem also holds for

- any fixed number field $F \subseteq \mathbb{R}$ in place of $\mathbb{Q}$ i.e. $S(F, H) \leq H^\varepsilon$;
- algebraic points whose coordinates have degree bounded by a fixed k, i.e. $S(k, H) \leq H^\varepsilon$, where $S(k, H) := |S \cap \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \text{for all } i, [\mathbb{Q}(x_i) : \mathbb{Q}] \leq k \text{ and } \operatorname{ht}(x_i) \leq H\}|$.

It is not possible to obtain an improvement in the H^ε bound which would hold for all o-minimal expansions of the real ordered field.

(Pila 1991) Given any function $\varepsilon(H) \to 0$ as $H \to \infty$, there is a transcendental analytic function $f\colon [0,1] \to \mathbb{R}$ and a sequence $(H_n)_n$ with $H_n \to \infty$ such that, for all $n \in \mathbb{N}, |\Gamma(f)(\mathbb{Q}, H_n)| \geq H_n^{\varepsilon(H_n)}$. These functions are definable in the o-minimal structure $\mathbb{R}_{\text{an}}$.

However, there is a proposed improvement for $\mathbb{R}_{\exp}$:

Wilkie's Conjecture (2006)

Let $F \subseteq \mathbb{R}$ be a number field of degree k. Suppose S is definable in $\mathbb{R}_{\exp}$ and does not contain an infinite semialgebraic subset. There exist $c(S,k), \gamma(S) > 0$ such that $|S(F,H)| \leq c(\log H)^\gamma$.

There is a version for algebraic points of bounded degree formulated by Pila (2010), where the exponent $\gamma = \gamma(S,k)$.

Pfaffian Functions I

In practice we consider a much wider class of functions than just the exponential function.
Such an approach is suggested by the following result:

Theorem (Pila 2007)

For any transcendental Pfaffian function $f: I \longrightarrow \mathbb{R}$ on an interval $I \subseteq \mathbb{R}$, there exist $c(f), \gamma(f) > 0$ such that $|\Gamma(f)(\mathbb{Q}, H)| \leq c(\log H)^{\gamma}$.

- (Khovanskii) *Pfaffian functions*: solutions to upper triangular systems of first order polynomial differential equations.
- They are analytic and o-minimal ($\mathbb{R}_{\text{Pfaff}}$, the expansion of the real ordered field by all Pfaffian functions, is o-minimal).
- They include $\exp$ on $\mathbb{R}$, $\log$ on $(0, \infty)$, $\sin$ on $(0, \pi)$, ...

(Khovanskii 1980) For a Pfaffian function f, we have effective bounds on the number of connected components of $Z(f), Z(f'), \ldots$, the zero sets of f and its derivatives, and on $\Gamma(f) \cap Z(P)$, the graph of f intersected with the zero set of a polynomial P,
which depend on the number of variables n of f, the *order* r of f (i.e. the number of functions defined by the system defining f), and the degree (α, β) of f, where α is a bound on the degrees of the polynomials in the system and β is the degree of the polynomial defining f, and (polynomially) on the degree d of P.

(Khovanskii 1980) If $g_1, \ldots, g_m \colon U \to \mathbb{R}$, $U \subseteq \mathbb{R}^n$, are Pfaffian of common order r and degree (α, β), and $P \in \mathbb{R}[X_1, \ldots, X_n]$ has degree d, then the number of connected components of $Z(g_1) \cap \ldots \cap Z(g_m) \cap Z(P)$ has an effective bound $c(n, r, \alpha, \beta) \cdot d^{n+r}$.

Implicit Definition

Adopting Pila's method for one-variable, transcendental Pfaffian functions, and first establishing the appropriate zero estimates for f implicitly defined from Pfaffian functions, we obtain:

Theorem (Jones-T. 2010)

Let $F \subseteq \mathbb{R}$ be a number field of degree k. Let $f : I \longrightarrow \mathbb{R}$ be transcendental and implicitly defined from Pfaffian functions, $I \subseteq \mathbb{R}$ an interval. There are $c(f,k), \gamma(f) > 0$ s.t. $|\Gamma(f)(F,H)| \leq c(\log H)^{\gamma}$.

Corollary (Jones-T. 2010)

For any $f : I \longrightarrow \mathbb{R}$ existentially definable in $\mathbb{R}_{Pfaff}$ s.t. $\Gamma(f)$ contains no infinite semialgebraic subset, there exist $c(f,k), \gamma(f) > 0$ s.t. $|\Gamma(f)(F,H)| \leq c(\log H)^{\gamma}$.

Proof.

By methods of Wilkie, such an f is piecewise implicitly defined. □

1-Dimensional Sets

Corollary (Jones-T. 2010)

For any $f : I \longrightarrow \mathbb{R}$ existentially definable in $\mathbb{R}_{\mathit{Pfaff}}$ s.t. $\Gamma(f)$ contains no infinite semialgebraic subset, there exist $c(f,k), \gamma(f) > 0$ s.t. $|\Gamma(f)(F,H)| \leq c(\log H)^{\gamma}$.

It follows directly that this bound will hold for any function definable in any model complete reduct of $\mathbb{R}_{\mathrm{Pfaff}}$ - in particular $\mathbb{R}_{\exp}$.

Corollary (Jones-T. 2010; also shown by Butler)

Wilkie's Conjecture holds for any 1-dimensional set S.

In each of these results, if we know the common complexity (m, r, α, β) of the Pfaffian functions implictly defining the curve, in each of the bounds we can compute the constant $c(m, r, \alpha, \beta, k)$ and the exponent $\gamma(m, r) = 3m + 3r + 8$.

Another approach: find suitable parameterizations in all dimensions, à la Pila–Wilkie. (Pila) This should be "mild parameterization": the covering functions ϕ are C^∞ and have roughly $||\phi^{(\alpha)}|| \leq |\alpha|^{C|\alpha|}$.

Theorem (Pila 2009)

Suppose that $S \subseteq \mathbb{R}^n$ has a mild parameterization. Then $S(F,H)$ is contained in at most $K_1(\log H)^{K_2}$ zero sets of polynomials of degree at most $(\log H)^{\frac{\dim S}{n-\dim S}}$, for some $K_1(S,k), K_2(S) > 0$.

Bound $|(S \cap Z(P))(F,H)|$, for a surface S, the graph of $f : U \longrightarrow \mathbb{R}$ implicitly defined from Pfaffian functions (again assume S contains no infinite semialgebraic subset):

Proposition (Jones-T. 2010)

For such a surface S, there exist $c(S,k), \gamma(S) > 0$ and a polynomial $Q \in \mathbb{R}[X]$ of degree $N(S)$ such that, for any polynomial $P : \mathbb{R}^3 \to \mathbb{R}$ of degree d, $|(S \cap Z(P))(F,H)| \leq cQ(d)(\log H)^\gamma$.

This proposition + all sets definable in $\mathbb{R}_{\text{an}}$ have a mild parameterization gives:

1st Corollary (Jones-T. 2010)

If $S \subseteq \mathbb{R}^n$ is a surface definable in $\mathbb{R}_{resPfaff}$, the real field expanded by all restricted Pfaffian functions, then there exist $c(S,k), \gamma(S) > 0$ such that $|S(F,T)| \leq c(\log H)^{\gamma}$.

2nd Corollary (Jones-T. 2010)

Wilkie's Conjecture holds for any surface S which admits a mild parameterization.

This question has also been addressed for special cases, and can also make sense for subsets of $\mathbb{C}$ (where $F \subseteq \mathbb{C}$).
In 2011, Masser proved the following bound for algebraic points of bounded degree on the graph of the Riemann zeta function:

Theorem (Masser 2011)

Let S be the graph of $\zeta\restriction_\Delta$, where $\Delta := \{z \mid |z - \frac{5}{2}| \leq \frac{1}{2}\}$. There exists an effective, absolute constant $C > 0$ such that, for all $H \geq e$,
$$|S(k,H)| \leq C\left(\frac{k^2 \log 4H}{\log(k \log 4H)}\right)^2.$$

Masser: "It may be an interesting problem to prove analogues ... for other natural functions. For example the Euler gamma function $\Gamma(z)$... Or the Weierstrass zeta function $\zeta_\Lambda(z)$ ... in spite of its differential equation we do not know a single rational z with $\zeta_\Lambda(z)$ irrational... Pila has also suggested $\frac{\zeta(z)}{\pi^z}$ [for ζ Riemann zeta]."

Theorem (Masser 2011)

Let S be the graph of $\zeta\restriction_\Delta$, where $\Delta := \{z \mid |z - \frac{5}{2}| \leq \frac{1}{2}\}$. There exists an effective, absolute constant $C > 0$ such that, for all $H \geq e$, $|S(k,H)| \leq C\left(\frac{k^2 \log 4H}{\log(k \log 4H)}\right)^2$.

(Boxall-Jones 2013) Analogous results hold for the functions $\Gamma(z)$ and $\frac{\zeta(z)}{\pi^z}$ (for ζ the Riemann zeta function), with exponent $3+\varepsilon$ in place of 2, holding for the restriction to $(2,\infty)$.

Independently, Besson (2013) adapted Masser's methods and proved that a bound

$$C(n)\frac{(k^2 \log H)^2}{\log(k \log H)}$$

holds for $\Gamma(z)\restriction_{[n-1,n]}$.

Let $\Lambda := \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\}$ be a lattice in the complex plane $\mathbb{C}$, with $|\omega_1| \leq |\omega_2|$.
The Weierstrass elliptic function $\wp_\Lambda : \mathbb{C} \setminus \Lambda \to \mathbb{C}$ corresponding to Λ is defined as follows:

$$\wp_\Lambda(z) := \frac{1}{z^2} + \sum_{n^2+m^2 \neq 0} \left(\frac{1}{(z + m\omega_1 + n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right).$$

We know

$$\begin{aligned} (\wp'_\Lambda(z))^2 &= 4(\wp_\Lambda(z))^3 - g_2\wp_\Lambda(z) - g_3 \\ &=: g_\Lambda(\wp_\Lambda(z)). \end{aligned}$$

By restricting to a fixed fundamental domain $\mathscr{F}$, we may define an inverse $(\wp_\Lambda)^{-1}$ which satisfies $(\wp_\Lambda^{-1})(z) = \int^z \frac{d\omega}{\sqrt{g_\Lambda(\omega)}}$.

Weierstrass ζ Functions

The Weierstrass zeta function $\zeta_\Lambda : \mathbb{C} \setminus \Lambda \to \mathbb{C}$ corresponding to Λ is defined by

$$\zeta_\Lambda(z) := \frac{1}{z} + \sum_{n^2+m^2 \neq 0} \left(\frac{1}{(z + m\omega_1 + n\omega_2)} + \frac{1}{(m\omega_1 + n\omega_2)} + \frac{z}{(m\omega_1 + n\omega_2)^2} \right).$$

It satisfies $\frac{\partial \zeta_\Lambda}{\partial z}(z) = -\wp_\Lambda(z)$.

We also know that ζ_Λ satisfies $\zeta_\Lambda(z) = -G_\Lambda(\wp_\Lambda(z))$, where

$$G_\Lambda(z) := \int^z \frac{\omega \, d\omega}{\sqrt{g_\Lambda(\omega)}}.$$

In order to prove his result, Masser establishes not only a good zero estimate for the Riemann zeta function, but also gives a new proof of the parameterization + diophantine part of the general strategy which applies to algebraic points of bounded degree in the complex setting.

This can be applied to the Weierstrass ζ_Λ function for a given lattice Λ to prove that $\Gamma(\zeta_\Lambda\restriction_{\Delta'})(k,H)$ lies in some $N(\Lambda)$ zero sets of polynomials of degree at most $c(k,\Lambda)\cdot\log H$, for some suitable disc Δ' depending on Λ.

The final step is to count the size of intersections $|\Gamma(\zeta_\Lambda)\cap Z(P)|$.

Recall: on a fundamental domain, $\zeta_\Lambda(z) = -G_\Lambda(\wp_\Lambda(z))$, where $G_\Lambda(z) := \int^z \frac{\omega\, d\omega}{\sqrt{g_\Lambda(\omega)}}$ and $(\wp_\Lambda^{-1})(z) = \int^z \frac{d\omega}{\sqrt{g_\Lambda(\omega)}}$.

Macintyre (2008): using Cauchy-Riemann equations, can obtain expressions for the 1st order derivatives of Re(G_Λ) and Im(G_Λ) (and Re$(\wp_\Lambda^{-1})$ and Im$(\wp_\Lambda^{-1})$) in terms of $\frac{\partial G_\Lambda}{\partial z}$ (respectively $\frac{\partial \wp_\Lambda^{-1}}{\partial z}$). For example,

$$\frac{\partial \mathrm{Re}(\wp_\Lambda^{-1})}{\partial x} = \frac{\mathrm{Re}(\sqrt{g_\Lambda(z)})}{|g_\Lambda(z)|} = \frac{\sqrt{\sqrt{A^2+B^2}+A}}{\sqrt{2(A^2+B^2)}},$$

where $g_\Lambda(z) = A(x,y) + iB(x,y)$. Hence, away from $\Lambda/2$ (the branch points of $\sqrt{g_\Lambda(z)}$), Re(G_Λ), Im(G_Λ), Re$(\wp_\Lambda^{-1})$, Im$(\wp_\Lambda^{-1})$ are real Pfaffian functions, and hence ζ_Λ is implicitly defined from Pfaffian functions of common order 9 and degree $(9,1)$. So if $\deg P \leq c \cdot \log H$, then $|\Gamma(\zeta_\Lambda) \cap Z(P)| \leq c' \cdot (\log H)^{15}$, and hence $|\Gamma(\zeta_\Lambda\restriction_{\Delta'})(k,H)| \leq c(k,\Lambda) \cdot (\log H)^{15}$.

Going back to an earlier result:

Theorem (Jones-T. 2010)

Let $F \subseteq \mathbb{R}$ be a number field of degree k. For any transcendental function $f : I \longrightarrow \mathbb{R}$ on an interval $I \subseteq \mathbb{R}$ implicitly defined from Pfaffian functions of common complexity (m, r, α, β), there exists $c(m, r, \alpha, \beta, k) > 0$ such that

$$|\Gamma(f)(F, H)| \leq c(\log H)^{3m+3r+8}.$$

This allows us to prove that for $\Gamma(\zeta_\Lambda \restriction_I)$, where $I \subseteq (\mathscr{F} \setminus \Lambda/2) \cap \mathbb{R}$ is an interval, and $F \subseteq \mathbb{C}$ a number field of degree $k \in \mathbb{N}$, there exists a constant $c(k) > 0$ such that $|\Gamma(\zeta_\Lambda \restriction_I)(F, H)| \leq c \cdot (\log H)^{41}$. Note that on the one hand the exponent is much worse (41 v. 15), but, on the other, the constant does not depend on the lattice Λ.