Model Theory and Harmonic Analysis on p-adic Groups
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1 History

In the late 1990s Denef and Loeser developed the theory of motivic integration. T. Hales and his students initiated a program to use motivic integration to study harmonic analysis on p-adic groups in a field-independent way. This is motivated by the following observation:

computing orbital integrals requires counting $\mathbb{F}_q$ points of hyperelliptic curves. A conceptual understanding of orbital integrals $\leftrightarrow$ associating right geometric objects to them. Denef and Loeser introduced model theory for motivic integration, which helped to accomplish this goal.

Most of the work we'll talk about today is due to Denef-Loeser-Cluckers (Motivic Integration) and Cluckers-Hales-Gordon-Halupczok (Applications)

2 Review of Harmonic Analysis on p-adic Groups

By p-adic groups we take to mean over a nonarchimedean local field.

Two aspects of harmonic analysis on p-adic groups are *orbital integrals* and *characters*, which play an important role in the trace formula.

Let G be finite and let $cl(G)$ the space of class functions on G, so that $cl(G) = \left\langle tr\pi \mid \pi \in \hat{G} \right\rangle = \left\langle Ch_{G_\gamma} \mid \gamma \in G/\sim \right\rangle$. For $f \in cl(G)$,

$\sum \mathcal{O}_\gamma(f) Ch_{G_\gamma} = f = \sum\limits_{\pi \in \hat{G}} G_\pi(f) tr(\pi)$ with $\mathcal{O}_\gamma(f), D_\pi(f) \in \mathbb{C}$ and $\mathcal{O}_\gamma(f) = f(\gamma) = \dfrac{1}{|G|} \sum_{g \in G} f(g\gamma g^{-1})$ and $D_\pi(f) =< f, tr\pi >$.

Now let $|G| = \infty$. An invariant distribution is given by a map $f \mapsto \mathcal{O}_\gamma(f) = \int\limits_{G/C_G(\gamma)} f(g\gamma g^{-1}) dg$.

If (π, V_π) is a representation of G, $f \in C_c^\infty(G)$ and $\pi(f) \in \text{End}(V_\pi)$ then $\pi(f)v = \int_F f(g)\pi(g)v dg$. Defin $\theta_\pi : C_c^\infty(G) \to \mathbb{C}$ by $\theta_\pi(f) = tr(\pi(f))$.

Simple case: Let $\Gamma \subseteq G$ be a unimodular group and consider $L^2(p\ G)$ with R a right representation of G acting on $L^2(p\ G)$. Then

$\sum\limits_{\gamma \in \Gamma/\sim} C_\gamma \mathcal{O}_\gamma(f) = Tr(R(f)) = \sum\limits_{\pi \in \hat{G}} a_\pi \theta_\pi(f)$

The side on the left is geometric; the side on the right is spectral.

3 Langlands functoriality

Langlands functoriality predicts that a natural correspondence between representations of two different groups by matching the spectral side of trace formula comes from matching geometric sides of the trace formula of respective groups.

Fundamental Lemma: Matching stable orbital integrals.

Ex. $\gamma_G \in G$, $\gamma_H \in H$; e.g. $H^\vee \to G^\vee$. $\mathcal{O}^{st}_{\gamma_G}(f) = \sum_{\gamma' \tilde{\gamma} stably conj.} \mathcal{O}_{\gamma'}(f)$

Let f_G, f_H be character functions of hyperspecial compact subgroups of G (resp H.)

The fundamental lemma approximately says that $\mathcal{O}_\gamma^{st} = \mathcal{O}_\gamma^{st}(f_H)$. It's been proved for $\mathbb{F}_q((t))$. Then by applying the transfer principle in model theory, for $p >> 0$ we can transfer from $\mathbb{F}_q((t))$ to F_ν for ν a nonarchimedean valuation. The proof for $\mathbb{F}_q((t))$ uses geometric tools from sheaf theory; then the fundamental lemma was transfered to the F_ν case by Cluckers-Loeser-Halupczok and in the case of the relative trace formula by Yim and Gordon.

We also transfer data from F_ν to $\mathbb{F}_q((t))$; the local integrability of characters and uniform boundedness of orbital integrals. The tools used to prove it in the F_ν case are the exponential map HA on the Lie algebra.

4 Model Theoretic component

We consider the three-sorted Denef-Pas language $\mathcal{L}_{DP} = \langle \mathcal{L}_{val}, \mathcal{L}_{res}, \mathcal{L}_{\mathbb{Z}} \rangle$ in sorts $(F, k_F, \mathbb{Z})$ together with maps from the sorts $ac : F \to k_F$ and ord : $F \to \mathbb{Z}$.

The definable sets include the (unramified) connected reductive group, most of May-Prasad filtration subalgebras, and the definable functions include elements of spherical Hecke algebras.

Motivic Exponential Sums

Linear combinations of π-definable functions with definable coefficients.

Motivic: Haar Measures, orbital integrals, Fourier transform of orbital integrals.

Uniform Boundedness Principle (C-G-H): Let $f : W \times \mathbb{Z}^n \to \mathbb{C}$ motivic with W Suppose that $|f(w\lambda)| \leq \alpha(\lambda)$. Then $|f(w\lambda) \leq q^{a+b||\lambda||}$ for $a, b \in \mathbb{Z}$ and $||\lambda|| = \sum_i |\lambda_i|$.

Corollary: Consider $|D(\gamma)|^{1/2}\mathcal{O}_\gamma(f_\lambda) \leq q^{a+b||\lambda||}$ for all semisimple γ where $f_\lambda = ch(K_G)\lambda(\overline{w}K_G)$ with λ a coroot (coming from the spherical Hecke algebras $X_* \cong \mathbb{Z}^{rkG}$), $D(\gamma)$ the Weyl discriminant .

Transfer principle: for $p >> 0$ then the truth of certain properties depend only on the residue field. Some of these properties include

- identity of motivic integrals (C-L)
- Uniform Boundedness Property (C-G-H)
- local integrability of θ_π. (C-G-H)

Questions: Understand θ_π. Result:

- $\pi = R_T^\theta$ of Deligne and Lusztig. Gordon showed that θ_π is motivic.
- For π the Steinberg representation. Then θ_π is motivic.

In general: we want an understanding of ramified group's multiplicative characters.