Uniformity in Representation Theory
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1 History

Let G be a finite group, $Irr(G) = \{\text{irreducible characters of } G\}$. To each such representation there is a homomorphism G to a matrix group.

Some facts from basic representation theory

$\sum_{\dim \chi \in Irr(G)} (\dim \chi)^2 = |G|$ which is the *second moment*

$|Irr(G)| = |\text{conjugacy classes of } G| = \sum_{\chi \in Irr(G)} (\dim \chi)^0$ which is the *zeroeth moment*

Theorem 1. *(Frobenius 1896) For every $k \geq 0$, the $-2k^{th}$ moment can be interpreted as $\sum_{Irr(G)} (\dim \chi) = \frac{|Hom(\pi_1 S_{k+1}, G)|}{|G|^{2k+1}}$ for S_{k+1} a surface of genus $g = k+1$.*

We will study the limiting behavior of $\mathrm{SL}_d(\mathbb{Z}/p^N)$ as $N \to \infty$ using this theorem. Its inverse limit is $\Gamma = \mathrm{SL}_d(\mathbb{Z}_p)$. The left hand side of the equation in Frobenius' theorem tends to $\sum_{Irr(\Gamma)} (\dim \chi)^{-2k}$ and the right converges to an integral of (the absolute value of a) top-degree form ω over a p-adic manifold $\int_{\mathrm{Hom}(\pi_1 S_{k+1}, \Gamma)|} |\omega|$.

Theorem 2. *For every $k \geq 11$, the $-2k^{th}$ moment can be interpreted as $\sum_{Irr(\Gamma)} (\dim \chi)^{-2k} = \int_{Hom(\pi_1 S_{k+1}, \Gamma)} |\omega|$.*

The same works for $\Delta := \ker(\mathrm{SL}_d(\mathbb{Z}_p) \to \mathrm{SL}_d(\mathbb{F}_p))$. Note that Δ is a pro-p group.

Theorem 3. *(Jaikin 2003) There are quantifier-free (in the Denef-Pas language) definable functions $f_1, f_2 : \mathbb{Z}_p^{d^2} \to \mathbb{Z}_p$ such that*

$$\zeta_\Delta(s) := \sum_{\chi \in Irr \Delta} (\dim \chi)^{-s} = \int_{\mathbb{Z}_p^{d^2}} |f_1| \cdot |f_2|^{-s}$$

for $Re(s) >> 0$ (just such that the sum converges.)

Remark: note that $\dim_{\mathrm{SL}_d(\mathbb{Z}_p)} = d^2$. Also, this tells us that there's a *definable* way to interpolate the data in the Frobenius theorem.

Corollary 1. *$\zeta_\Delta(s)$ as a rational function in p^s, and we can meromorphically continue $\zeta_\Delta(0)$, $\zeta_\Delta(-2)$.*

Theorem 4. *(Jaikin, Klopsch) $\zeta_\Gamma(-2) = 0$.*

Question: Is there a meaning to $\zeta_\Gamma(0)$? We know it's a nonzero number.

Frobenius' formula holds for the group $\mathrm{SL}_d(\mathbb{F}_p[[t]])$. The transfer principle tells you that $\zeta_{\mathrm{SL}_d(\mathbb{Z}_p)}(2k) = \zeta_{\mathrm{SL}_d(\mathbb{F}_p[[t]])}(2k)$ if $p >> k$.

The theorem if Jaikin uses the exponential map very crucially, so you can't use that in the characteristic p case; instead, use transfer to get it for large enough p.

Conjecture 1. *If $p >> 0$, then $\zeta_{SL_d(\mathbb{Z}_p)}(s) = \zeta_{SL_d(\mathbb{F}_p[[t]])(s)}$ independently of k. Equivalently,*

$|n-$*dimensional irreducible characters of* $SL_d(\mathbb{F}_p[[t]])| = |n-$*dimensional irreducible characters of* $SL_d(\mathbb{Z}_p)|$

Equivalently, if you look at the group of locally constant functions to $\mathbb{C}$, $(\zeta(SL_d(\mathbb{Z}_p)), *) = (\zeta(SL_d(\mathbb{F}_p[[t]])), *)$.

Evidence:

1. True for SL_2
2. True for units in quaternion algebra