The Geometry of Algebraic Dynamics
Presentation by Holly Krieger

1 Motivating Question

What geometric conditions on an algebraic variety affect the existence of interesting dynamical systems on that variety?

What do we mean by *interesting*? One place to look is the dynamical Manin-Mumford.

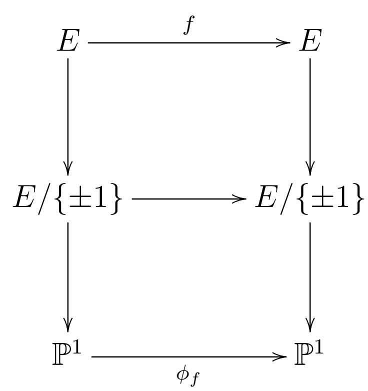
The setup:

- A dominant endomorphism $f : X \to X$ with X smooth, projective, over $\mathbb{C}$
- Forward orbits: let $\alpha \in X$ and define the forward orbit of f on α as $\mathcal{O}_f(\alpha) = \{f^{\circ n}(\alpha)\}_{n\in\mathbb{N}}$
- fixed ($f(Y) = Y$), periodic ($f^n(Y) = Y$), and preperiodic ($f^n(Y) = f^m(Y)$ with $m \neq n$) subvarieties

To have interesting cases, we need to assume that f has infinite order.

Example: Curves

- genus 0: on $\mathbb{P}^1 \to \mathbb{P}^1$ there are many such interesting families, ex. $f(z) = z^d + c$
- genus 1 (elliptic curves): multiplication and CM give many infinite order endomorphisms. Also, there are the Lattès maps $f : E \to E$ which make the following diagram commute



- genus $g(C) \geq 2$. There is a theorem [de Franchis]: there are only finitely many surjective $\phi : C \to C$. Thus, very dynamically uninteresting.

Definition 1. Assume that X is a smooth projective variety over $\mathbb{C}$. The *canonical divisor* of X is the divisor:

$$K_X = \text{zeros of } \omega - \text{poles of } \omega$$

for any meromorphic top form ω. The multiples of nK_X determine rational $\phi_{nK_X} : X \to \mathbb{P}^n$. The *Kodaira dimension* of X is the maximal dimension of $\phi_{nK_X}(X)$ for $n \in \mathbb{N}$

1. If this map is undefined (when $H^0(X, nK_X) = \{0\}$ for all $n \in \mathbb{N}$) we say $\kappa_X = -\infty$.

2. $\kappa_X \in \{-\infty, 0, 1, \cdots, \dim X\}$
3. $\kappa_{\mathbb{P}^1} = -\infty$, $\kappa_E = 0$, $\kappa_C = 1$ if $g(C) \geq 2$ (so that Kodaira dimension distinguishes these three classes of curves)
4. $\kappa_{X \times Y} = \kappa_X + \kappa_Y$
5. finite surjective morphisms cannot increase Kodaira dimension

Theorem 1 (Kobayashi-Ochiai)**.** *If X has $\kappa_X = \dim X$ (i.e. "of general type") then X carries only finitely many endomorphisms.*

This theorem rules out dynamical properties of very many algebraic varieties.

Definition 2. A map $\phi : X \to X$ is **polarizable** if there is an ample line bundle $\mathcal{L}$ and $q > 1$ such that the pullback of $\mathcal{L}$, $\phi^*\mathcal{L} \cong \mathcal{L}^{\otimes q}$, then $(X, \phi, \mathcal{L})$ is a polarized dynamical system.

- A degree $d > 1$ morphism $\mathbb{P}^n \to \mathbb{P}^n$ is a polarizable dynamical system.
- Multiplication $[n] \in \text{End}(A)$, $n \neq 0, \pm 1$ on an abelian variety forms a polarizable dynamical system.
- A map $\phi : \mathbb{P}^1 \times E$ is *not* polarizable

1. This definition guarantees that $|\{x \mid f^m(x) = f^n(x)\} < \infty$ is finite for each m, n.

Fakhouddin This condition is equivalent to there being a commutative diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle i} \\ \mathbb{P}^N & \xrightarrow[\Phi]{} & \mathbb{P}^N \end{array}$$

2. This result allows for a canonical height of $(X, \mathcal{L}, \phi)$ which can allow us to characterize what we mean by a special point, allows for equidistribution to make sense. The height is

$$\hat{h}_{\phi,\mathcal{L}} : X \to \mathbb{R}_{\geq 0}$$

such that $\hat{h}_{\phi,\mathcal{L}}(x) = 0$ if and only if x is preperiodic.

2 Dynamical Manin-Mumford Conjecture

Principle: If $Y \subseteq X$ is a closed irreducible subvariety then Y is special if and only if Y has a Zariski-dense subset of special points.

Conjecture 1 (Zhang)**.** *For special = preperiodic, this holds for polarized dynamical systems.*

This is false [Ghioca-Tucker-Zhang]. To see this, let E be a CM curve, $|\omega_1| = |\omega_2| > 1$ and consider $[\omega_1] \times [\omega_2] : E \times E \to E \times E$. If $\dfrac{\omega_1}{\omega_2}$ is not a root of 1, $\Delta_{E\times E}$ is not perperiodic but Prep$(E \times E) =$ Tors$(E \times E)$.

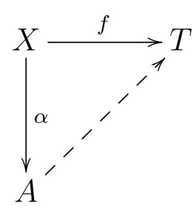
Modification: Require a Zariski dense subset of preperiodic points on which the induced map on the $Gr_{\dim Y}(T_{X,x})$ leaves $T_{Y,x}$ preperiodic. This has been verified for abelian varieties and lines in $\mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 2 (Falkhouddin)**.** *Let X be smooth projective variety over $\mathbb{C}$ such that $\kappa_X \geq 0$. Then if $(X, \mathcal{L}, \phi)$ is polarized then $X \cong A/G$ with A an abelian variety and G a finite group acting fixed-point freely.*

Corollary 1. *(Does not require smoothness) If $(X, \mathcal{L}, \phi)$ is polarized then $\kappa_X \leq 0$.*

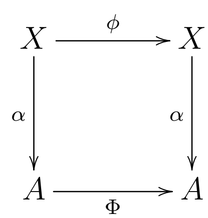
This theorem rules out $E \times C$ from having polarized dynamical structures for E elliptic and C of genus ≥ 2 as $\kappa_{E\times C} = 1$.

Fact From any X smooth projective over $\mathbb{C}$ there is an $\alpha : X \to A$ an abelian variety (the Albanese variety of X) which is unique to translation, satisfying the universal property



for complex tori T.

This induces a dynamical system on the Albanese variety given by the commutative diagram



Theorem 3 (Serre + Lefschetz)**.** *If ϕ is polarized then $\Phi : A \to A$ is without loss of generality an isogeny.*

Theorem 4 (Pink-Roessler, Krieger-Reschke)**.** *Let $\Phi : A \to A$ be an isogeny. Let u_Φ be the number of root of unity eigenvalues on $H^{1,0}(A)$. Then if V is an irreducible subvariety of A with $\Phi(V) = V$, $\kappa(V) \leq u_\Phi$,*

Corollary 2. *If $(X, \mathcal{L}, \phi)$ is polarized then the induced map from the Albanese variety is surjective.*