Galois covers in positive characteristic

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Connections for Women:
Model Theory and its interactions with Number Theory and Arithmetic Geometry
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Abstract

Title: Galois covers in positive characteristic

There are some interactions between model theory and Artin-Schreier extensions of fields.

I will discuss wildly ramified covers of curves in positive characteristic p.

The presence of an automorphism of order p leads to significant constraints on arithmetic invariants of the Jacobian.

I will also discuss results about deformations of wildly ramified covers of curves in positive characteristic.

Motivation - Model theory and Galois theory

Manin 2012
Combinatorial cubic surfaces and reconstruction theorems.
Reconstruct field of definition and equation of a projective cubic surface V from the set of its K-points $V(K)$.

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Bertrand 2008
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The author can only thank (resp. apologize to) the organizers of the Newton conference for helping him realize (resp. becoming aware so late of) the relevance of model theory to this circle of problems.

Motivation - Model theory and Galois Theory

Pop 2012
Lifting of Curves - the Oort conjecture

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Maybe it's interesting to mention that the first variant of the proof was shorter, but relied heavily on model theoretical tools and was not effective (concerning the finite extensions of the Witt vectors over which the smooth lifting can be realized).

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Outline:
1) Function fields in positive characteristic (tame v. wild)
2) Artin-Schreier extensions
3) Deformation of wildly ramified covers of curves
4) Structure of Hurwitz space of wildly ramified covers

Some model theory remarks: undecidability

Church: first order logic is undecidable - no effective method for determining membership.

First order logic can distinguish between:

Poonen/Pop: function fields over large fields of positive char. from 5 other classes of fields.

Tsen/Lang: function fields over k of distinct transcendence degrees.

Duret: two non-isomorphic function fields of curves over $k = \overline{k}$, $\mathrm{char}(k) > 0$. (see also Chermin and Pierce).

So...there exists set of first order sentences satisfied by function fields of genus g with given invariants, e.g., p-rank.

Tame versus wild: part A

Let k be an algebraically closed field.
Let X be a smooth projective connected k-curve of genus g.
Let B be a non-empty finite set of points of X (genus g).

If $\mathrm{char}(k) = 0$, then $\pi_1(X - B)$ is a free group of rank $2g + |B| - 1$. Example: $\pi_1(\mathbb{A}^1_{\mathbb{C}}) = 0$.

If $\mathrm{char}(k) = p$, then this is **false**.

The fundamental group in char. p

If $X - B$ is an affine k-curve, the algebraic fundamental group $\pi_1(X - B, x_\circ)$ is infinitely profinitely generated.

The structure of $\pi_1(X - B, x_\circ)$ is not known for any affine k-curve $X - B$, although its finite quotients have been determined (Harbater and Raynaud).

Tame versus wild: part B

Let $\phi : Y \to X$ be a (Galois) cover of k-curves.

If $\mathrm{char}(k) = 0$, then inertia groups are cyclic $I \simeq \mathbb{Z}/m$.
Riemann-Hurwitz: $\mathrm{genus}(Y)$ depends only on $\deg(\phi)$, $\mathrm{genus}(X)$, and orders of inertia groups.

If $\mathrm{char}(k) = p$, then this is **false**.

Inertia groups in char. p

ϕ is *wildly ramified* if p divides the order of an inertia group I.
If so, $I \simeq P \rtimes \mathbb{Z}/m$ with P a p-group and $p \nmid m$.

The genus of Y depends on extra ramification information, filtrations of higher ramification groups.
Model theory: Epp eliminating wild ramification; Kulmann local uniformization.

Tame versus wild: part C

If $\mathrm{char}(k) = 0$, Riemann's Existence Theorem:
Given X, B, and G, there are finitely many G-Galois covers $\phi : Y \to X$ branched only at B.

If $\mathrm{char}(k) = p$, then this is **false**.

Moduli of covers in char. p

There are non-isotrivial deformations of almost all wildly ramified covers $\phi : Y \to X$ for which G, B, the inertia groups, and the ramification filtrations are constant.

To illustrate new phenomena, use Artin-Schreier extensions: $K \subset L$ given by equation of form $y^p - y = h$ for some $h \in K$.

Tame versus wild: part D

Let J_X be the Jacobian of X (p.p. abelian variety dim. g).

If $\mathrm{char}(k) = 0$, then there are p^{2g} points of order p in $J_X(k)$.

If $\mathrm{char}(k) = p$, then this is **false**.

The p-rank in char. p

If X is a k-curve of genus g, the number of points of order p on the Jacobian $J_X(k)$ equals p^s for some integer s such that $0 \leq s \leq g$. Here f is called the *p-rank* of X.

Example for $g = 1$: elliptic curve is ordinary if $f = 1$ and supersingular if $f = 0$.

Artin-Schreier curves

Let $k = \overline{k}$ with $\mathrm{char}(k) = p$.

Definition

An Artin-Schreier curve is a (smooth, projective connected) curve X/k with $\mathbb{Z}/p \subset \mathrm{Aut}(X)$ s.t. the quotient is $\mathbb{P}^1_k$.

Then X has an equation $y^p - y = h(x)$ for some $h(x) \in k(x)$.

The $\mathbb{Z}/p$-cover $\phi : X \to \mathbb{P}^1_k$ is given by the field extension $k(x) \hookrightarrow k(x)[y]/(y^p - y - h(x))$.

Automorphism of order p given by $\sigma(x) = x$, $\sigma(y) = y + 1$.

Automorphism respects equation since binomial coefficient $\binom{p}{i} \equiv 0 \bmod p$ for $1 \le i \le p - 1$.

Ramification

Artin-Schreier theory: can modify $h(x)$ by $z^p - z$ for any $z \in k(x)$ without changing isomorphism class of ϕ.
WLOG: orders of poles of $h(x)$ are rel. prime to p.

Branch locus

The cover $\phi : X \to \mathbb{P}^1_k$ is branched at the poles of $h(x)$.

Suppose $h(x)$ has pole of order d_i at P_i for $1 \leq i \leq r+1$,

i.e., $\mathrm{div}_\infty(h(x)) = \sum_{i=1}^{r+1} d_i P_i$ with $p \nmid d_j$.

Let $e_i = d_i + 1$. Then $e_i \not\equiv 1 \bmod p$.

The branch divisor of ϕ is $D = \sum_{i=1}^{r+1} e_i P_i$.

The genus

Definition

The genus of X is $\dim(H^0(X,\Omega^1))$ (v.s. of holo. 1-forms)

Recall: $h(x) \in k(x)$ has pole order $d_i = e_i - 1$ at P_i and $p \nmid d_i$.

Wild Riemann-Hurwitz formula

The genus of Artin-Schreier curve X with equation $y^p - y = h(x)$ is $g = d(p-1)/2$ where $d + 2 = \sum_{i=1}^{r+1} e_i$.

An Artin-Schreier curve of genus $g = d(p-1)/2$ can be classified by partition $\vec{E}$ of $d+2$ s.t. if $e \in \vec{E}$ then $e \not\equiv 1 \bmod p$.

The p-rank

The p-rank is an invariant of X with wide significance.
Invariant of both Newton polygon of X and p-torsion group scheme of $\mathrm{Jac}(X)$.
Measures number of roots of L-polynomial of X (or exponential function of $h(x)$) having p-adic absolute value 1.

Def: the p-rank s of X is the integer in $\{0, \ldots, g\}$ so that the number of p-torsion points on the Jacobian of X is p^s.

Deuring-Shafarevic formula

The p-rank of Artin-Schreier curve X is $s = r(p-1)$ where $r+1$ is the number of branch points of $\phi : X \to \mathbb{P}^1_k$.

An Artin-Schreier curve of genus $g = d(p-1)/2$ and p-rank $s = r(p-1)$ can be classified by a partition $\vec{E} = \{e_1, \ldots, e_{r+1}\}$ of $d+2$ s.t. each $e_i \not\equiv 1 \bmod p$.

Example: new phenomena

Let $G = \mathbb{Z}/p$, $X \simeq \mathbb{P}^1_k$, $B = \{\infty\}$.
Let $h(x) \in k[x]$ have degree j where $p \nmid j$.

Consider AS cover $\phi : X \to \mathbb{P}^1_k$ with equation $y^p - y = h(x)$.

Wild B: the ramification filtration becomes trivial at index j.
$\mathrm{genus}(X) = (j-1)(p-1)/2$ can be arbitrarily large.
Wild A: $\pi_1(\mathbb{A}^1_k)$ infinitely profinitely generated.
Wild C: If $j \geq 3$, varying $h(x)$ yields a non-isotrivial deformation of ϕ.
Wild D: the p-rank of X is $s = 0$.

Similar results are true for G-Galois covers of any affine k-curve, as long as p divides $|G|$, but their proofs require advanced techniques when G is not a p-group.

Existence of Artin-Schreier curves

An Artin-Schreier curve of genus $g = d(p-1)/2$ and p-rank $s = r(p-1)$ can be classified by a partition $\vec{E} = \{e_1, \ldots, e_{r+1}\}$ of $d+2$ s.t. each $e_i \not\equiv 1 \bmod p$.

Example ($p = 2$)

Let $g \geq 0$ and $0 \leq s \leq g$. Then $2g+2$ can be partitioned into $s+1$ even integers. So, there exists an Artin-Schreier curve of genus g and p-rank s in characteristic 2.

Example ($p \geq 3$)

There exists an Artin-Schreier k-curve of genus $g = d(p-1)/2$ with p-rank 0 iff $p \nmid (d+1)$.
There exists an Artin-Schreier k-curve of genus $g = d(p-1)/2$ with p-rank g (ordinary) if and only if $2 \mid d$.

Moduli spaces for Artin-Schreier curves

Consider the moduli spaces:
$\mathcal{AS}_g$ of Artin-Schreier curves of genus g.
$\mathcal{AS}_{g,s}$ of Artin-Schreier curves of genus g and p-rank s.
$\mathcal{AS}_{g,\vec{E}}$ of Artin-Schreier curves with partition $\vec{E}$.

Maugeais: $\mathcal{AS}_g$ is an algebraic stack,
whose irreducible components have dimension $\frac{2g-(p-1)}{p-1}$.

Two natural questions are:
1) Is $\mathcal{AS}_g$ irreducible? how many irreducible components?
2) What is the dimension of $\mathcal{AS}_{g,\vec{E}}$?

Partitions with congruence condition

Let Ω_d be the set of partitions of $d+2$ into positive integers $e_1, e_2, \ldots$ with each $e_i \not\equiv 1 \bmod p$.

Let $\Omega_{d,r} \subset \Omega_d$ consist of partitions of length $r+1$.

There is a natural partial ordering $<$ on Ω_d so that $\vec{E} < \vec{E}'$ if $\vec{E}'$ is a refinement of $\vec{E}$.

This yields a directed graph G_d: vertices correspond to partitions $\vec{E}$ in Ω_d; there is an edge from $\vec{E}$ to $\vec{E}'$ if and only if $\vec{E} \lneq \vec{E}'$ and there is no partition lying in between them.

An edge $\vec{E} < \vec{E}'$ in G_d can be of two types.

Example: $p = 2$ and $d = 10$ (so $g = 5$)

$$\begin{tikzcd}
 & & & \{12\} \arrow[dll] \arrow[d] \arrow[drr] & & & \\
 & \{2,10\} \arrow[dl] \arrow[drr] & & \{4,8\} \arrow[dlll] \arrow[d] \arrow[drrr] & & \{6,6\} \arrow[dll] & \\
\{2,2,8\} \arrow[drr] & & & \{2,4,6\} \arrow[dl] \arrow[dr] & & & \{4,4,4\} \arrow[dll] \\
 & & \{2,2,2,6\} \arrow[dr] & & \{2,2,4,4\} \arrow[dl] & & \\
 & & & \{2,2,2,2,4\} \arrow[d] & & & \\
 & & & \{2,2,2,2,2,2\} & & &
\end{tikzcd}$$

Example: $p = 3$ and $d = 10$ (so $g = 10$)

$$
\begin{tikzcd}
 & & \{12\} \arrow[dll] \arrow[dd] \arrow[dr] \arrow[drr] & & \\
 & & & \{6,6\} \arrow[ddll, bend right] \arrow[dr] & \{3,9\} \arrow[d] \\
\{2,2,8\} \arrow[dr] \arrow[drr] & & \{2,5,5\} \arrow[d] & & \{3,3,6\} \arrow[ddll] \arrow[d] \\
 & \{2,2,2,6\} \arrow[dr] \arrow[dd] & \{2,2,3,5\} \arrow[d] & & \{3,3,3,3\} \\
 & & \{2,2,2,3,3\} & & \\
 & \{2,2,2,2,2,2\} & & &
\end{tikzcd}
$$

Main result

Theorem: P/Zhu

Let $g = d(p-1)/2$ with $d \geq 1$ and $s = r(p-1)$ with $r \geq 0$.

- The irreducible components of $\mathcal{AS}_{g,s}$ are exactly the moduli spaces $\mathcal{AS}_{g,\vec{E}}$ with $\vec{E} \in \Omega_{d,r}$.
- If $\vec{E} = \{e_1, \ldots, e_{r+1}\} \in \Omega_{d,r}$, then the dimension of $\mathcal{AS}_{g,\vec{E}}$ is $d - 1 - \sum_{i=1}^{r+1} \lfloor (e_i - 1)/p \rfloor$ over k.

Corollary

The moduli space $\mathcal{AS}_g$ is irreducible if and only if: (i) $p = 2$; (ii) $p = 3$ and $g = 1, 2, 3, 5$; or (iii) $p \geq 5$ and $g = 0$ or $g = (p-1)/2$.

Example: $p = 2$ and $d = 10$ (so $g = 5$)

Notation: partition $\vec{E}$ - dimension of $\mathcal{AS}_{g,\vec{E}}$

$$
\begin{tikzcd}
 & & & \{12\}-4 \arrow[dll] \arrow[d] \arrow[drr] & & & \\
 & \{2,10\}-5 \arrow[dl] \arrow[drr] & & \{4,8\}-5 \arrow[dlll] \arrow[d] \arrow[drrr] & & \{6,6\}-5 \arrow[dll] \arrow[dr] & \\
\{2,2,8\}-6 \arrow[dr] & & & \{2,4,6\}-6 \arrow[dll] \arrow[drr] & & & \{4,4,4\}-6 \arrow[dl] \\
 & \{2,2,2,6\}-7 \arrow[drr] & & & & \{2,2,4,4\}-7 \arrow[dll] & \\
 & & & \{2,2,2,2,4\}-8 \arrow[d] & & & \\
 & & & \{2,2,2,2,2,2\}-9 & & &
\end{tikzcd}
$$

When $p = 2$

Corollary: Let $p = 2$. Let $g \geq 1$. Let $0 \leq s \leq g$.

- The irreducible components of $\mathcal{AS}_{g,s}$ are in bijection with partitions of $g+1$ of length $s+1$.
- Every component has dimension $g-1+s$.

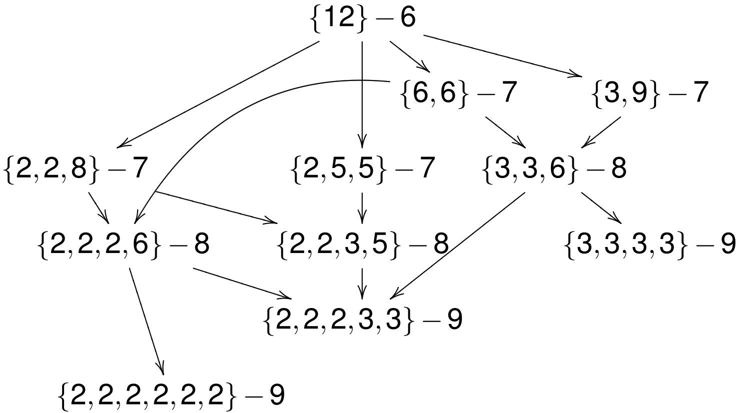
The second statement is a transversality result: the condition that X is an Artin-Schreier curve in char. 2 is 'independent' from the condition that X has 2-rank s.

When $p = 2$, we also prove that $\mathcal{AS}_{g,\vec{E}_1}$ is in the closure of $\mathcal{AS}_{g,\vec{E}_2}$ if and only if $\vec{E}_1 < \vec{E}_2$ (deformation result).

In particular, $\mathcal{AS}_g$ is irreducible when $p = 2$.

Example: $p = 3$ and $d = 10$ (so $g = 10$)

Notation: partition $\vec{E}$- dimension of $\mathcal{AS}_{g,\vec{E}}$



Ideas from the proof

We add data to moduli problem $\mathcal{AS}cov_{g,\vec{E}}$,
(fix $\iota : \mathbb{Z}/p \hookrightarrow \mathrm{Aut}(X)$ and $X/\langle\iota\rangle \simeq \mathbb{P}^1$)

We prove there is a morphism $\mathcal{AS}cov_{g,\vec{E}} \to Cdiv_{d+2,\vec{E}}$,
where the 'configution space' $Cdiv_{d+2}$ denotes relative effective Cartier divisors of degree $d+2$.
After finite flat extension, $D = \sum_{i=1}^{r+1} e_i P_i$.

The image of C is irreducible of dimension $r+1$.

Each fiber of C is isomorphic to a quotient of a product of affine spaces.
Rely on Krull dimension calculation of Bertin and Mezard.

Example: $p = 5$ and $d = 10$ (so $g = 20$)

Portion of graph shows geometry of $\mathcal{AS}_g$ is complicated.
Notation: partition $\vec{E}$- dimension of $\mathcal{AS}_{g,\vec{E}}$

$$
\begin{tikzcd}
 & & \{12\}-7 \arrow[dl] \arrow[dr] & \\
 & \{2,10\}-8 \arrow[dl] & & \{3,9\}-8 \arrow[dd] \\
\{2,2,8\}-8 \arrow[d] & & & \\
\{2,2,4,4\}-9 \arrow[dr] & & & \{3,3,3,3\}-9 \\
 & \{2,2,2,2,4\}-9 \arrow[dr] & & \\
 & & \{2,2,2,2,2,2\}-9 &
\end{tikzcd}
$$

Deformation Question

Remark:

If $p \geq 5$ and $g \geq p-1$, then there exist partitions $\vec{E}_1 < \vec{E}_2$ such that $\mathcal{AS}_{g,\vec{E}_1}$ is NOT in the closure of $\mathcal{AS}_{g,\vec{E}_2}$.

Theorem: P/Zhu

Let $0 \leq s < g$. If a component of $\mathcal{AS}_{g,s}$ is not open/dense in irreducible component of $\mathcal{AS}_g$ then it is in the closure of a component of $\mathcal{AS}_{g,s+p-1}$.

Key topic: deformation of widely ramified cover of curves with non-constant branch locus.

Deformation proof

Tool: formal patching - reduce to $\hat{\phi} : \hat{Y} \to \mathrm{Spec}(k[[x^{-1}]])$ cover of complete local rings of curves at ramification point. Equation for $\hat{\phi}$ is $y^p - y = x^{e-1}$.

Let $e = e_1 + e_2$ with $p \mid e_1$.
Want to increase number of branch points.
Deform over $\mathrm{Spec}(k[[t]])$ to $y^p - y = x^{e-1}/(1 - xt)^{e_1}$.

Pole at $x^{-1} = 0$: ram. jump equals order of pole $e_2 - 1$.
Pole at $x = 1/t$: by AS theory, ram. jump is not e_1 but $e_1 - 1$.

Flatness: $e = e_1 + e_2$ (ram. divisor same degree).

Arbitrary affine curve and Galois group

Fix X/k, B, and G a finite quotient of $\pi_1(X-B)$.

Consider $\phi : Y \to X$, group G, branch B.

If ϕ is wildly ramified,

- what possibilities for g_Y occur?
- what are ramification filtrations and conductors?

Method: Use existence of one cover to produce another with larger genus.

Techniques: formal patching and 'twist' of Galois action.

Inertia of local cover

Let $\phi : Y \to X$ be a G-Galois cover branched at b.

Let I be the inertia group of ϕ at b. Suppose $p \mid |I|$.

Let $\hat{\phi} : \hat{Y} \to \hat{X}$ be the germ of ϕ,

(I-Galois cover of $\mathrm{Spec}(k[[x]])$ branched at closed point).

Note $\hat{\phi}$ is totally ramified.

Also $I \simeq P \rtimes \mathbb{Z}/m$ where $|P| = p^e$ and $p \nmid m$.

Artin representation

Given $\hat{\phi} : \hat{Y} \to \hat{X}$, group I, totally ramified.

Let t be a uniformizer at the unique ramification point.

For $s \in I$, $s \neq 1$, let $i(s) = \mathrm{val}(s(t) - t)$.

Define character $a_{\hat{\phi}}(s) = \begin{cases} -i(s) & s \neq 1 \\ \sum_{s \neq 1} i(s) & s = 1 \end{cases}$.

This is character of Artin representation for $\hat{\phi}$.

The conductor

The function $i(s) = \mathrm{val}(s(t) - t)$ determines:

* a filtration of I (upper numbering);

* the different of $\hat{\phi}$;

* the local contribution to g_Y.

The *conductor* $c_{\hat{\phi}}$ is the rational number where filtration dies.

Theorem

Let $k = \overline{k}$ with $\mathrm{char}(k) = p$.

Theorem: P

Let X be smooth projective k-curve.

Let B be finite nonempty set of points. Let $N \in \mathbb{N}$.

Let G be a finite quotient of $\pi_1(X - B)$ s.t. p divides $|G|$.

Then $\exists\ \phi : Y \to X$, group G, branch B, with $g_Y > N$.

Corollary

If $X = \mathbb{P}^1_k$ and $B = \{\infty\}$, and $c \in \mathbb{N}$ suff. large with $p \nmid c$

Then $\exists\ \phi : Y \to X$, group G, branch B, conductor c.

Outline of proof

1. Show $\exists$ wildly ramified $\phi_1 : Y_1 \to X$, group G, branch B.
2. Analyze $\hat{\phi}_1 : \hat{Y}_1 \to \hat{X}$, group I, conductor c_1.
3. Formal patching: can deform ϕ_1 to increase genus if can deform $\hat{\phi}_1$ to increase c_1.

   (flat deformation, but special fiber reducible, ϕ is restriction to one component)
4. Twist $\hat{\phi}_1$ to increase c_1.

Proof: Twisting action

Given I-Galois cover $\hat{\phi} : \hat{Y} \to \hat{X}$ with conductor c.

Let $A \subset I$ be last non-trivial ramification group.

Then A central in I; let $\bar{I} = I/A$.

Let $\bar{\phi}$ be the $\bar{I}$-Galois quotient of $\hat{\phi}$.

Let $\hat{\pi} = \pi_1(\mathrm{Spec}(k((x^{-1})))).$

I-Galois covers $\hat{\phi}$ are in bijection with $\mathrm{Hom}(\hat{\pi}, I)$.

Twisting Galois covers

Let H be the fibre of $\mathrm{Hom}(\hat{\pi}, I) \to \mathrm{Hom}(\hat{\pi}, I/A)$ over $\overline{\phi}$.

The cover $\hat{\phi}$ corresponds to $\hat{\rho} : \hat{\pi} \to I$ in fiber.

Theorem (P): H is principal homogeneous space for $\mathrm{Hom}(\hat{\pi}, A)$.

$A \simeq (\mathbb{Z}/p)^a$, so A-Galois covers are easy to understand. Given $\alpha : \hat{\pi} \to A$ (an A-cover).

Twist $\hat{\rho}$ by α to get $\hat{\rho}^\alpha : \hat{\pi} \to I$

where $\hat{\rho}^\alpha(\omega) = \hat{\rho}(\omega)\alpha(\omega)\ \forall\ \omega \in \hat{\pi}$.

The twist is another I-Galois cover $\tilde{\phi}$ which dominates $\overline{\phi}$.

Effect on conductor

Let H be fibre of $\mathrm{Hom}(\hat{\pi}, I) \to \mathrm{Hom}(\hat{\pi}, I/A)$ over $\overline{\phi}$.

Given $\alpha : \hat{\pi} \to A$ (an A-cover).

Twist $\hat{\rho}$ by α to get $\hat{\rho}^{\alpha} : \hat{\pi} \to I$

Twisting changes only tail of filtration.

Prop. Usually, $\mathrm{cond}(\hat{\rho}^{\alpha}) = \max\{\mathrm{cond}(\hat{\rho}), \mathrm{cond}(\alpha)\}$.

Example: equations for twist

Given $\hat{\phi} : \hat{Y} \to \hat{X}$, group I, conductor c.

Look at $\hat{Y} \xrightarrow{A} \overline{Y} \xrightarrow{I/A} \hat{X}$.

Given $\psi_\alpha : \hat{Z} \to \hat{X}$, group $\mathbb{Z}/p$.

Twist $\hat{\phi}$ to $\hat{\phi}^\alpha : \hat{Y}^\alpha \to \hat{X}$.

$$\text{Equation for} \begin{cases} \hat{Y} \to \overline{Y} & y^p - y = r_1 \in K(\overline{Y}) \\ \hat{Z} \to \hat{X} & z^p - z = r_2 \in K(\hat{X}) \\ \hat{Y}^\alpha \to \overline{Y} & y^p - y = r_1 + r_2 \end{cases}$$

This preserves I-Galois action.

Hard to compute conductor using equations.