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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Alex Kruckman **Email/Phone:** Kruckman@gmail.com

Speaker's Name: Antoine Chambert-Loir

Talk Title: Specialities for non-specialists (I)

Date: 02/03/14 **Time:** 9:30 am / pm (circle one)

List 6-12 key words for the talk: Transcendental number theory, Diophantine equations, elliptic curves, complex tori.

Please summarize the lecture in 5 or fewer sentences: Part 1 of 3. An introduction to the theory of special points on varieties. Motivation from transcendental number theory (exponentiation, the modular j-function, elliptic curves, complex tori, Gauss's hypergeometric function) and diophantine equations.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- [x] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [x] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- [x] For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- [x] When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- [x] Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Goal: If varieties are "special," they have a lot of "special" points, whatever "special" means.

1^{st} motivation: transcendental number theory

The exponential function.

Lindemann's theorem: If $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$, then $\exp(\alpha)$ is transcendental.

In particular, e is transcendental

π is transcendental (since $e^{i\pi} = -1$)

The modular j-function

Let H be the Poincaré upper half plane $\{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$

The group $SL(2,\mathbb{R})$ acts by homographis on H.

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad z \in H, \quad \gamma \cdot z = \frac{az+b}{cz+d}$$

The modular function is the unique holomorphic function $j: H \to \mathbb{C}$ such that $j(\gamma \cdot z) = j(z)$ $\forall \gamma \in SL(2,\mathbb{Z})$

and, when $\operatorname{Im}(z) \to +\infty$,

$$j(z) = \frac{1}{q} + 744 + \sum_{n=1}^{\infty} c_n q^n \quad (c_n \in \mathbb{Z})$$

$$q = \exp(2\pi i z)$$

$j(i) = 1728$,

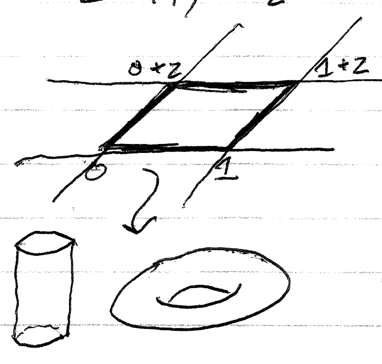
$$j\left(\frac{1-i\sqrt{3}}{2}\right) = 0, \quad j\left(\frac{1+i\sqrt{163}}{2}\right) = -(640320)^3$$

Schneider (1937) If $z \in \overline{\mathbb{Q}} \cap H$ and $j(z) \in \overline{\mathbb{Q}}$, then z is imaginary quadratic.

Complex multiplication - the converse is true.

Elliptic Curves

$z \in H$, $L_z = \mathbb{Z} + z\mathbb{Z}$, $A_z = \mathbb{C}/L_z$.



Function theory on A_z

$\leftrightarrow$ meromorphic functions on $\mathbb{C}$ with periods 1 and z.

Weierstrass $\wp$ $\quad \wp(u) = \dfrac{1}{u^2} + \displaystyle\sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \left[\frac{1}{(u-(m+nz))^2} - \frac{1}{(m+nz)^2}\right]$

$\wp$ is periodic. Functional equation: $\wp'(u)^2 = 4\wp(u)^3 - g_2(z)\wp(u) - g_3(z)$

$\boxed{\text{Parametrization of the algebraic curve with eq. } y^2 = 4x^3 - g_2x - g_3}$

$$g_2(z) = 60 \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m+nz)^4}$$

$$g_3(z) = 140 \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m+nz)^6}$$

$$j(z) = 1728 \frac{g_2(z)^3}{g_2(z)^3 - 27 g_3(z)^2}$$

Conversely, every (complex) elliptic curve is obtained in this way for a single value of $z \in H$ up to the action of $SL(2,\mathbb{Z})$)

$j(z) \in \overline{\mathbb{Q}} \longleftrightarrow$ the elliptic curve A_z has a plane cubic equation with coefficients in $\overline{\mathbb{Q}}$.

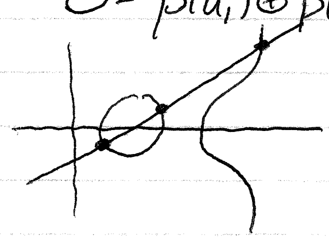
z imaginary quadratic $\longleftrightarrow$???

A_z is a complex Lie group described by the addition formulae for the $\wp$ function.

$$p : \mathbb{C}\setminus L_z \longrightarrow \mathbb{C}^2 \qquad u \longmapsto (\wp(u), \wp'(u))$$

$$p : A_z \longrightarrow \mathbb{P}^2(\mathbb{C})$$
$$\mathbb{C}\setminus L_z \ni u \longmapsto [\wp(u) : \wp'(u) : 1]$$
$$L_z \ni u \longmapsto [1:0:0] = \underline{\underline{O}}$$

$O = p(u_1) \oplus p(u_2) \oplus p(u_3) \longleftrightarrow u_1 + u_2 + u_3 \in L_z$
$\longleftrightarrow p(u_1), p(u_2), p(u_3)$ lie on a line



$\mathrm{End}(A_z) = \mathbb{Z}$, usually

But it can be bigger.

$$z = \frac{c + dz}{a + bz}$$

$\Rightarrow$ z is quadratic if $a \notin \mathbb{Z}$

$$\begin{tikzcd} \mathbb{C} \arrow[r, "u \mapsto \alpha u"] \arrow[d] & \mathbb{C} \arrow[d] \\ \mathbb{C}/L_z \arrow[r] & \mathbb{C}/L_z \end{tikzcd}$$

One needs $u \in L_z$

$\Rightarrow \alpha u \in L_z$

$\underline{u = 1}$: $\alpha \in L_z$: $\alpha = a + bz$

$\underline{u = z}$: $z\alpha \in L_z$: $z\alpha = c + dz$

Generalization to higher dimensions

complex tori $\quad \mathbb{C}^g/L$

$(g \geq 1) \quad L \subset \mathbb{C}^g$ lattice

(Un)fortunately, there may be no nonconstant meromorphic L-periodic functions on $\mathbb{C}^g$.

Riemann gave the conditions for the existence of sufficiently many L-periodic meromorphic functions on $\mathbb{C}^g$ so that $\mathbb{C}^g/L$ embeds as an algebraic subvariety of some projective space.

Condition: There exists a positive definite Hermitian form (h) on $\mathbb{C}^g$ such that $\mathrm{Im}(h)$ takes integral values on L.

$\rightsquigarrow L \sim \mathbb{Z}^g + Z \cdot \mathbb{Z}^g$, where $Z \in M_g(\mathbb{C})$, symmetric, $Z = X + iY$, Y positive definite.

$\mathcal{H}_g = \{X + iY \in M_g(\mathbb{C}),\ \text{symmetric},\ Y > 0\}$
~ Siegel upper half space.

Analogue of the j-function:

H. Cartan: $\quad J : \mathcal{H}_g \longrightarrow \mathbb{P}^N(\mathbb{C})$ invariant under $Sp(2g, \mathbb{Z})$

$J(Z) = J(Z') \leftrightarrow A_Z \simeq A_{Z'}$

The image A_g is the moduli space of (p.p.) abelian varieties

Shimura: One can normalize J so that A_g is defined over $\overline{\mathbb{Q}}$.

Cohen/Shiga-Wolfart: Z is in $\overline{\mathbb{Q}}$ and $J(Z)$ is in $\overline{\mathbb{Q}}$

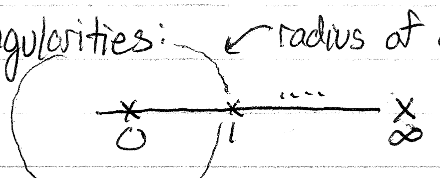
$\leftrightarrow A_Z$ has sufficiently many complex multiplications

Gauss's hypergeometric function:

$$F(a,b,c;z) = 1 + \frac{ab}{c} z + \frac{a(a+1)b(b+1)}{2!\,c(c+1)} z^2 + \ldots$$

Satisfies $$z(1-z)F'' + (c - (a+b+1)z)F' - abF = 0$$

Singularities: radius of convergence just the unit disk



There is monodromy as soon as you go around 1 or ∞ (in analytic continuation)

Monodromy group $\Delta \subset GL(2,\mathbb{C})$

Δ can be finite – $F_{a,b,c}$ is algebraic $\left(z\in\overline{\mathbb{Q}} \Rightarrow F(z)\in\overline{\mathbb{Q}}\right)$

Δ can be infinite

$\boxed{\begin{array}{l} a,b,c\in\mathbb{Q} \\ c<1 \\ 0<a<c \\ 0<b<c \end{array}}$

$\nearrow$ Δ is "arithmetic" (There are infinitely many $z\in\overline{\mathbb{Q}}$ s.t. $F(z)\in\overline{\mathbb{Q}}$)

$\searrow$ Δ is "non-arithmetic" (The set of $z\in\overline{\mathbb{Q}}$ s.t. $F(z)\in\overline{\mathbb{Q}}$ is finite)

$z \rightsquigarrow$ alg. curves $C_z : y^N = x^u(1-x)^v(z-x)^w$
C_0

$$F(z) = \frac{\int_{\gamma_z}\omega_z}{\int_{\gamma_0}\omega_0}$$ γ_z/γ_0 ~~curves~~ paths on C_z/C_0

When is this quotient of integrals algebraic?

$\rightsquigarrow$ abelian varieties $Jac(C_z)$, $Jac(C_0)$

Wüstholz: $F(z)\in\overline{\mathbb{Q}} \longleftrightarrow Jac(C_z)$ is related to $Jac(C_0)$

André-Oort (Edixhoven-Yafaev)

2nd motivation: Diophantine equations

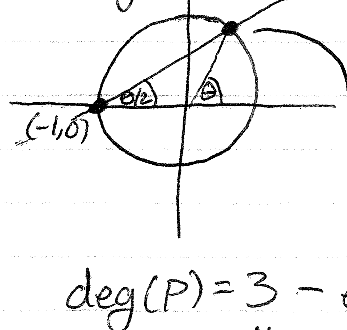
simplest cases: one equation in two variables, $P(x,y)=0$

$\deg(P)=1$ – line – one rational solution $\Rightarrow$ infinitely many

$\deg(P)=2$ – conic — $x^2+y^2=-1$ no solutions

$x^2+y^2=3$ no solutions (reduce mod 4)

$x^2+y^2=1$ draw a circle



rational slope $t=\tan(\theta/2)$

$x=\frac{1-t^2}{1+t^2}$ $y=\frac{2t}{1+t^2}$ (many rational points)

$\deg(P)=3$ – elliptic curve – one gets a group law on the rational points of the elliptic curve

Mordell (1922) E elliptic$/\mathbb{Q}$

The group $E(\mathbb{Q})$ of rational points is finitely generated.

Weil (1928) Extension – to number fields

– to abelian varieties

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<u>Next case</u>: $\deg(P) > 3$ (and some non-degeneracy conditions)
Conjecture of Mordell: There are finitely many rational pts.
<u>Faltings</u> (1983) This is true. (Very nonconstructive)