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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Alex Kruckman **Email/Phone:** Kruckman@gmail.com

Speaker's Name: Pierre Simon

Talk Title: An Introduction to Stability-Theoretic Techniques (I)

Date: 02/03/14 **Time:** 11:00 (am) pm (circle one)

List 6-12 key words for the talk: Model theory, elementary equivalence, types, quantifier elimination, elimination of imaginaries

Please summarize the lecture in 5 or fewer sentences: Part 1 of 3. Slides with supporting boardwork. An overview of the basic definitions and tools in model theory. Compactness, Lowenheim-Skolem, monster models, types, quantifier elimination, and elimination of imaginaries.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- [x] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [x] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- [x] For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
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Examples of basic definitions

Structure $(K, +, -, \cdot, 0, 1)$

Formula $\varphi = \forall x \exists y \; y \cdot y = x$

We write $M \models \varphi$ if M satisfies φ

If a formula has free variables, i.e. $\psi(x) = \exists y \; y \cdot y = x$,

We write $M \models \psi(a)$ if $a \in M$ satisfies ψ

$\bar{x} = x_1, x_2, x_3$ (for example) is a tuple of variables.

Then $|\bar{x}| = 3$, the length of the tuple.

Given $\psi(x)$, $\psi(M)$ is the <u>definable set</u> in M defined by ψ.

Usually <u>definable</u> means <u>definable with parameters</u>.

Given $\theta(\bar{x}, \bar{y})$ and $\bar{b} \in M^{|\bar{y}|}$, we can use $\bar{b}$ as parameters, $\theta(\bar{x}, \bar{b})$.

Question from audience: Is a formula the same thing as a sentence?

Answer: A sentence is a formula with no free variables.

Obtaining uniform bounds from compactness

Theory T, $f(M) \in \mathbb{R}_{\geq 0}$ s.t. $f(M) > 0$ always.

Say $f(M) \geq 1/n \longleftrightarrow M \models \varphi_n$.

$T' = T \cup \{\neg \varphi_n \mid n \in \mathbb{N}\}$ is inconsistent

$\Rightarrow \exists n$ s.t. $f(M) \geq 1/n$ always.

Fields of different characteristics are not elementarily equivalent:

One satisfies $\underbrace{1 + 1 + \dots + 1}_{p \text{ times}} = 0$, the other does not.

$M \preceq N \Rightarrow M \equiv N$ (take φ with no free variables in the definition of $\preceq$)

Converse is not true.

Example: $M = (\mathbb{N}, \leq)$, $N = (\{-1\} \cup \mathbb{N}, \leq)$. $M \simeq N$, so $M \equiv N$.

$M \subseteq N$, but $M \not\preceq N$. Let $\varphi(x)$: $\exists y \; y < x$, $a = 0$.

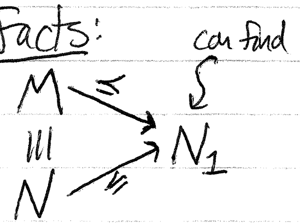
$M \models \neg \varphi(a)$, but $N \models \varphi(a)$.

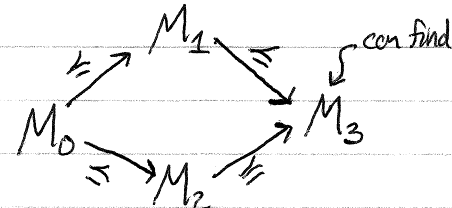
But $\overline{\mathbb{Q}} \preceq \mathbb{C}$.

— Start with $\overline{\mathbb{Q}}$. Let $\kappa = 2^{\aleph_0} = |\mathbb{C}|$.
$LS \Rightarrow \exists K,\ \overline{\mathbb{Q}} \preceq K,\ |K| = \kappa$.
$\text{tr.deg}(K) = 2^{\aleph_0} \Rightarrow K \cong \mathbb{C}$.

One can do the same with any two alg. closed fields of the same characteristic.

<u>Two facts:</u>

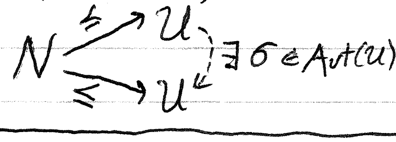




Also the union of a chain of elementary extensions is an elementary extension.

Put together by amalgamation
<u>all models</u> of a complete theory $T = Th(M)$
— too many! (arbitrarily large cardinality)

Fix κ very big. There is a structure $\mathcal{U} \succeq M$ s.t.
(1) For any $N \equiv M$, $|N| < \kappa$, $\exists j_N : N \preceq \mathcal{U}$
(2) If $|N| < \kappa$,



<u>2 perspectives on types</u>
- A set of formulas (everything true about a tuple)
- An orbit under an automorphism group of the monster model
  $tp(\bar{a}/B) = tp(\bar{b}/B)$ iff $\exists \sigma \in Aut(\mathcal{U})$, σ fixes B, $\sigma(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$

In the type space $S_{\bar{x}}(B)$, $[\varphi(\bar{x}, \bar{b})] = \{ p \mid \varphi(\bar{x}, \bar{b}) \in p \}$
is a basic <u>clopen</u> set: $(\bar{b} \in B)$
$[\varphi(\bar{x}, \bar{b})]^c = [\neg \varphi(\bar{x}, \bar{b})]$.

<u>Example</u>: $M = \overline{\mathbb{Q}} \preceq \mathcal{U}\ (= \mathbb{C})$. $S_{x_1,x_2}(\overline{\mathbb{Q}}) = \{ tp(a_1, a_2 / \overline{\mathbb{Q}}) \mid a_1, a_2 \in \mathcal{U} \}$
$p = tp(a_1, a_2 / \overline{\mathbb{Q}})$.
<u>Case 1</u>: $\text{trdeg}(a_1, a_2 / \overline{\mathbb{Q}}) = 0$.
Then $a_1, a_2 \in \overline{\mathbb{Q}}$, $(x_1 = a_1) \wedge (x_2 = a_2) \in p$ and completely determines p.

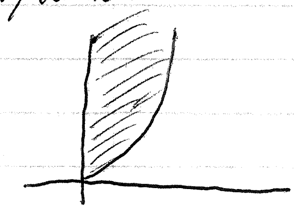
<u>Case 2</u>: $\mathrm{trdeg}(a_1, a_2/\overline{\mathbb{Q}}) = 2$. No polynomial is 0 on a_1, a_2.
This also completely determines p (any pair of tr.deg. 2 is in the orbit of any other under automorphisms of $\mathcal{U}$)

<u>Case 3</u>: $\mathrm{trdeg}(a_1, a_2/\overline{\mathbb{Q}}) = 1$.
Then $C(a_1, a_2) = 0$ for some $C(x_1, x_2) \in \overline{\mathbb{Q}}[x_1, x_2]$.
(a_1, a_2) lies on the curve C.
But $x_1 = c \notin p$ for all $c \in \overline{\mathbb{Q}}$.

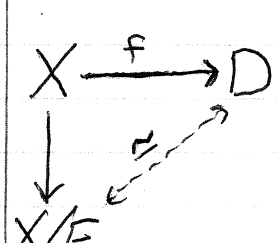
Topologically, <u>Case 1</u> is isolated, the other two are not (you need to say infinitely many things).

<u>Quantifier elimination</u>: For all $\varphi(\bar{x})$, there is $\psi(\bar{x})$ with no quantifiers s.t. $T \vdash \forall \bar{x}\ \varphi(\bar{x}) \leftrightarrow \psi(\bar{x})$.

Error on slide: $\varphi(x)$ should be $\varphi(x): \exists y\ (y^2 = x)$
$\varphi(\mathbb{R}) = \mathbb{R}_{\geq 0}$ - not the set defined by any polynomial equation or inequation.
Add the order to the language. $\leq$ is definable:
$x \leq y \leftrightarrow \exists z\ (z^2 = y - x)$
so there are no new definable sets.
Now we see that the definable sets are the semialgebraic sets ↗



<u>Elimination of imaginaries</u>



Given E a definable equivalence relation, $\exists$ definable D and definable $f: X \to D$ s.t. $f(x) = f(y) \Leftrightarrow x E y$.

<u>To code definable sets</u>: $\varphi(\bar{x}, \bar{y})$ a formula, one defines $E_\varphi(\bar{y}, \bar{y}')$: $E_\varphi(\bar{b}, \bar{b}')$ iff $\forall \bar{x}\ \varphi(\bar{x}, \bar{b}) \leftrightarrow \varphi(\bar{x}, \bar{b}')$
$\ulcorner \varphi(\bar{x}, \bar{b}) \urcorner = \bar{b}/E_\varphi$ is the code of $\varphi(\bar{x}, \bar{b})$.
↖ canonical element coding the definable set

Question: What is $\bar{b}/E_\varphi$? Answer: It is $f(\bar{b})$, the class of $\bar{b}$ under E_φ.

Question: Can you always add something to a structure so it has quantifier elimination?

Answer: Given a formula $\varphi(\bar{x})$, add a relation $R_\varphi(\bar{x})$ to the language, and set $R_\varphi(\bar{a})$ iff $M \models \varphi(\bar{a})$.

This is not very instructive.

Introduction to model theoretic techniques

Pierre Simon

Université Lyon 1, CNRS and MSRI

Introductory Workshop: Model Theory, Arithmetic Geometry and Number Theory

Basic definitions

- structure $\mathcal{M} = (M; R_1, R_2, ..., f_1, f_2, ..., c_1, c_2, ...)$
- formula $\exists x \forall y R_1(x, y) \vee \neg R_2(x, y)$
- language L
- satisfaction $\mathcal{M} \models \varphi$
  $\bar{a} \in M, \quad \mathcal{M} \models \psi(\bar{a})$
- theory T: (consistent) set of sentences
- model $\mathcal{M} \models T$
- definable set $\psi(\bar{x}) \longrightarrow \psi(M) = \{\bar{a} \in M^{|\bar{x}|} : \mathcal{M} \models \psi(\bar{a})\}$

  usually definable set means definable *with parameters*:
  $\theta(\bar{x}; \bar{b}) \longrightarrow \theta(M; \bar{b}) = \{\bar{a} \in M^{|\bar{x}|} : \mathcal{M} \models \theta(\bar{a}; \bar{b})\}$

Compactness

Theorem (Compactness theorem)

If all finite subsets of T are consistent, then T is consistent.

Uses of compactness

- Transfer from finite to infinite.
- From infinite to finite:
  Approximate subgroups (see tutorial on multiplicative combinatiorics)
  Szemeredí's theorem (Elek-Szegedy, Towsner ...)
- Obtaining uniform bounds

Understanding definable sets of M

$Th(M)$: set of sentences true in the structure M.

- Elementary equivalence: $M \equiv N$ if $Th(M) = Th(N)$.

  Example: If K and L are two algebraically closed fields of the same characteristic, then $K \equiv L$.
- A theory T is *complete* if it is of the form $Th(M)$.
- Elementary extension: $M \preceq N$ if $M \subseteq N$ and for all $\varphi(\bar{x})$ and $\bar{a} \in M$, $M \models \varphi(\bar{a}) \iff N \models \varphi(\bar{a})$.

Theorem (Löwenheim-Skolem)

Assume that L is countable, M infinite.

- *Let $\kappa \geq |M|$, then there is an elementary extension $M \prec N$, where $|N| = \kappa$.*
- *If $A \subseteq M$, then there is $M_0 \preceq M$ containing A, $|M_0| = |A| + \aleph_0$.*

- Monster model $\mathcal{U}, \mathbb{C}, \mathbb{M}, \ldots$

Types

Let $B \subset M$ and $\bar{a} \in M^k$.

Definition

The *type* of $\bar{a}$ over B is the set of formulas

$$\{\varphi(\bar{x}; \bar{b}) : \bar{b} \in B^{|\bar{b}|}, M \models \varphi(\bar{a}; \bar{b})\}.$$

Fact

The tuples $\bar{a}, \bar{b} \in \mathcal{U}^k$ have the same type over $B \subset \mathcal{U}$ iff there is an automorphism $\sigma : \mathcal{U} \to \mathcal{U}$ fixing B pointwise such that $\sigma(\bar{a}) = \bar{b}$.

The set of types over B (in a given variable $\bar{x}$) is denoted by $S_{\bar{x}}(B)$. It is a totally disconnected compact space.

Quantifier elimination

Definition

A theory T *eliminates quantifiers* in a language L if every L-formula is equivalent modulo T to a formula without quantifiers.

Examples:

- $Th(\mathbb{C}; 0, 1, +, -, *)$ eliminates quantifiers;
- $Th(\mathbb{R}; 0, 1, +, -, *)$ **does not** eliminate quantifiers:

$$\varphi(x) \equiv \exists y (x^2 = y)$$

- $Th(\mathbb{R}; 0, 1, +, -, *, \leq)$ eliminates quantifiers.

If T eliminates quantifiers and $M, N \models T$, then

$$M \subseteq N \Longrightarrow M \preceq N.$$

Examples

- $(\mathbb{N}; \leq)$;
- $(\mathbb{C}; 0, 1, +, -, *)$;
- $(\mathbb{R}; 0, 1, +, -, *, \leq)$.

Imaginaries

Let $X \subseteq M^k$ be a definable set and $E \subseteq X^2$ a definable equivalence relation. Then X/E is an *imaginary sort* of M.

We say that M *eliminates imaginaries* if every imaginary sort is definably isomorphic to a definable set.

Examples: $\mathbb{C}$, $\mathbb{R}$ eliminate imaginaries.

Codes of definable sets

End of talk 1.