MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Alex Kruckman **Email/Phone:** Kruckman@gmail.com

Speaker's Name: Deirdre Haskell

Talk Title: A model-theorist's view of algebraically closed valued fields

Date: 02 / 03 / 14 **Time:** 2 : 00 am / (pm) (circle one)

List 6-12 key words for the talk: ACVF, swiss cheeses, elimination of imaginaries, stable domination

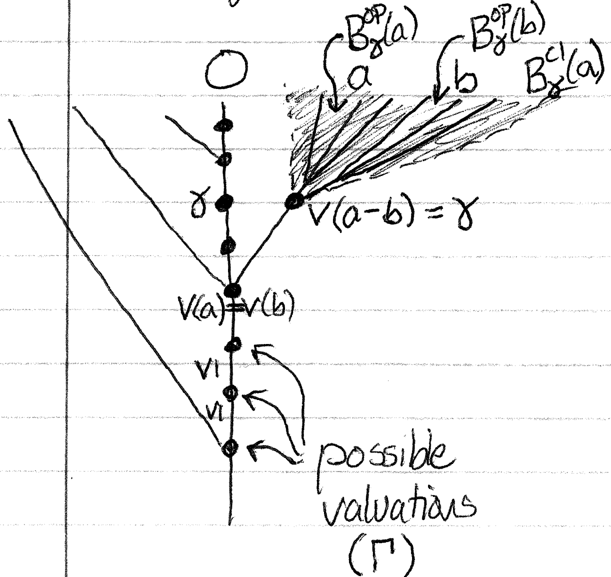
Please summarize the lecture in 5 or fewer sentences: Slides, with two illustrations on the board. After defining valued fields and giving examples, the speaker introduced the theory ACVF of algebraically closed valued fields. A description of the definable sets (via quantifier elimination), types, and imaginaries was given. The end of the talk introduced the stable part of a model of ACVF, stable domination (with examples), and orthogonality.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- [x] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [x] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- [x] For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- [x] When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- [x] Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Picturing a valued field



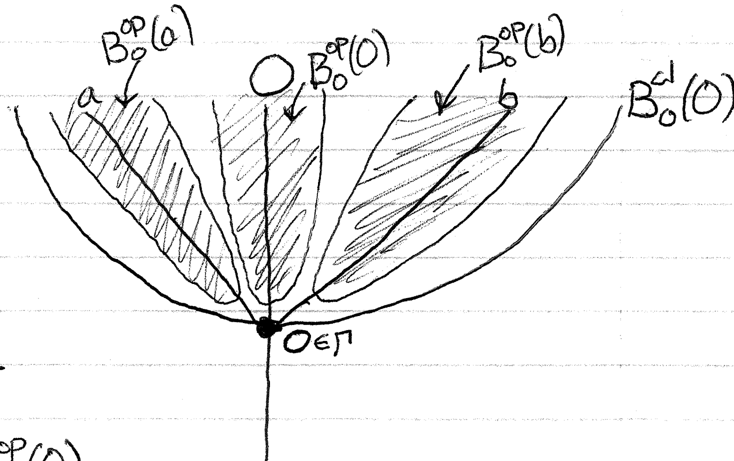
$a \neq 0$

$b \neq a$, but $v(b) = v(a)$

$B^{cl}_\gamma(a) = \{x \mid v(x-a) \geq \gamma\}$

$B^{cl}_\gamma(b) = B^{cl}_\gamma(a)$

$B^{op}_\gamma(b) \neq B^{op}_\gamma(a)$



$B_0^{cl}(0) = \{x \mid v(x) \geq 0\} = \mathcal{O}_K$
$B_0^{op}(0) = \{x \mid v(x) > 0\} = m_K$

residue field $k = B_0^{cl}(0) / B_0^{op}(0)$

= open balls at $\mathcal{O}$

A model-theorist's view of algebraically closed valued fields

Deirdre Haskell

McMaster University

MSRI program on model theory, number theory and arithmetic geometry
Introductory Workshop
Berkeley CA USA, 3–7 February 2014

valued field, algebraically closed

Field K, with valuation function $v : K^\times \to \Gamma$, an ordered group satisfying:

$$\begin{aligned} \forall x, y \in K \quad v(xy) &= v(x) + v(y) \\ v(x+y) &\geq \min\{v(x), v(y)\}. \end{aligned}$$

For convenience, we often write $v(0) = \infty$.
We will write

$$\begin{aligned} \mathcal{O}_K &= \{x \in K : v(x) \geq 0\}, \\ \mathfrak{m}_K &= \{x \in K : v(x) > 0\}, \\ k &= \mathcal{O}_K / \mathfrak{m}_K. \end{aligned}$$

Given another valued field L, we will refer to $\Gamma_L, \mathcal{O}_L, \mathfrak{m}_L, k_L$.

valued field, algebraically closed

A valued field K may or may not be algebraically closed. If it is, then k is also algebraically closed and Γ_K is divisible.

Examples

$\bigcup_{n=1}^{\infty} \mathbb{C}((X^{1/n}))$ is algebraically closed. Here $\Gamma = \mathbb{Q}$, $k = \mathbb{C}$, characteristic is $(0,0)$.

$\bigcup_{n=1}^{\infty} \widetilde{\mathbb{F}}_p((X^{1/n}))$ is henselian, but not algebraically closed. Here $\Gamma = \mathbb{Q}$, $k = \tilde{\mathbb{F}}_p$, characteristic is (p,p).

$\widetilde{\mathbb{Q}}_p$. Here $\Gamma = \mathbb{Q}$, $k = \tilde{\mathbb{F}}_p$, characteristic is $(0,p)$.

ACVF

We work in a sorted language, with sorts for the field, the residue field and the value group.

The language $\mathcal{L}_{k\Gamma}$

On the field sort K we have the language of rings $+, \cdot, 0, 1$
On the residue field sort k we have the language of rings $+, \cdot, 0, 1$
On the value group sort Γ we have the language of ordered groups $+, <, 0, \infty$
Between the sorts we have two functions $\mathrm{Res} : K \times K \to k$, $v : K \to \Gamma \cup \infty$

The theory of valued fields

- K is a field
- k is a field
- Γ is an ordered group, ∞ is larger than everything else
- v is a non-trivial valuation
- $\mathrm{Res}(x, y) = \mathrm{res}(x/y)$, if $v(x/y) \geq 0$ and $\mathrm{Res}(x, y) = 0 \in k$, if $v(x/y) < 0$.

ACVF

The theory of algebraically closed valued fields

Add a family of axioms to say that every polynomial in one variable has a root in the field.

This is not a complete theory; to make it complete, have to specify the characteristic (p, q).
From now on, take K to be a model of ACVF, $K \models ACVF$.

definable sets in ACVF — balls

A definable set is $\{x : \varphi(x, \bar{a})\}$, where φ is a formula in the language, $\bar{a}$ is a tuple of parameters, x is a (tuple of) variables in any of the sorts.

closed ball radius γ around a

$$\{x \in K : v(x - a) \geq \gamma\} = B_{\gamma}^{cl}(a)$$

We allow a singleton to be a closed ball $B_{\infty}^{cl}(a)$.

open ball radius γ around a

$$\{x \in K : v(x - a) > \gamma\} = B_{\gamma}^{op}(a)$$

We allow all of K as an open ball $B_{-\infty}^{op}(a)$.

The open balls form a basis for a topology on K called the valuation topology. Open and closed balls are both clopen in the sense of the topology.

definable sets in ACVF — balls

Of course, $B_0^{cl}(0) = \mathcal{O}_K$ and $B_0^{op}(0) = \mathfrak{m}_K$.

Moreover, for any a, γ, there is a 1-1 correspondence between $B_\gamma^{cl}(a)$ and $\mathcal{O}_K$, and between $B_\gamma^{op}(a)$ and $\mathfrak{m}_K$ given by $x \to (x-a)/c$, where $v(c) = \gamma$. Neither a nor c is unique.

$B_\gamma^{cl}(0)$, $B_\gamma^{op}(0)$ are $\mathcal{O}_K$-modules, and the quotient $B_\gamma^{cl}(0)/B_\gamma^{op}(0)$ is a one-dimensional k-vector space and is isomorphic to k.

We can write

$$\begin{aligned} B_\gamma^{cl}(0)/B_\gamma^{op}(0) &= \{a + B_\gamma^{op}(0) : v(a) = \gamma\} \\ &= \{B_\gamma^{op}(a) : v(a) = \gamma\} \end{aligned}$$

definable sets in ACVF — swiss cheeses

Since two balls are either nested or disjoint, the intersection of finitely many balls is just a ball. Thus boolean combinations of balls can be written in a canonical way as unions of *swiss cheeses*

$$B \setminus C_1 \cup \cdots \cup C_n,$$

where B, $C_1, \ldots, C_n$ are balls.

definable sets in ACVF — swiss cheeses

Since two balls are either nested or disjoint, the intersection of finitely many balls is just a ball. Thus boolean combinations of balls can be written in a canonical way as unions of *swiss cheeses*

$$B \setminus C_1 \cup \cdots \cup C_n,$$

where B, $C_1, \ldots, C_n$ are balls.

Because the field is algebraically closed, $\{x \in K : v(p(x)) \geq 0\}$, where p is a polynomial in x, is also a finite union of swiss cheeses. Thus every quantifier-free definable set in one variable is a finite union of swiss cheeses.

definable sets in ACVF — swiss cheeses

Since two balls are either nested or disjoint, the intersection of finitely many balls is just a ball. Thus boolean combinations of balls can be written in a canonical way as unions of *swiss cheeses*

$$B \setminus C_1 \cup \cdots \cup C_n,$$

where B, $C_1, \ldots, C_n$ are balls.

Because the field is algebraically closed, $\{x \in K : v(p(x)) \geq 0\}$, where p is a polynomial in x, is also a finite union of swiss cheeses. Thus every quantifier-free definable set in one variable is a finite union of swiss cheeses.

Theorem

Every formula in the language is equivalent in ACVF to a formula without quantifiers.

Corollary

Every definable set in one variable is either finite or has non-empty interior in the valuation topology.

types

A *type* p in one variable over parameters C is a complete consistent set of formulas $\{\varphi(x, \bar{c})$ with $c \in C$. The set of realizations of p in K is

$$\{a \in K : \varphi(a, \bar{c}) \text{ for all } \varphi \in p\}.$$

Thus a type is an infinite intersection of definable sets.

Examples

Fix $K \models ACVF$, element $a \notin K$.

$$\mathrm{tp}(a/K) = \{\varphi(x, \bar{c}) : \varphi(a, \bar{c}) \text{ holds}\}.$$

By quantifier elimination and the swiss cheese decomposition, we may assume that each formula $\varphi(x, \bar{c})$ says either "x is in a given ball" or "x is not in a given ball".

Examples: a is generic in a closed ball

The type says that a is in the closed ball $B_\gamma^{cl}(c)$ and a is not in any smaller ball defined over K.
Then $v(a-c) \geq \gamma$, but $v(a-c) \not> \gamma$, so $v(a-c) = \gamma$; in fact, $v(a) = v(c)$.
But $B_\gamma^{op}(a)$ is not defined over K, so $\mathrm{res}(a/c) \notin k_K$.
Thus $K(a)$ is a residual extension of K.

This is a *definable type*: for any formula $\psi(x, y)$, there is a formula $d\psi(y)$ such that $\psi(x, c) \in \mathrm{tp}(a/C)$ iff $d\psi(c)$ holds.

types in ACVF

Examples: a is generic in an open ball

The type says that a is in the open ball $B_\gamma^{op}(c)$ and a is not in any smaller ball defined over K.

Then $v(a-c) > \gamma$, but $v(a-c) \not\geq \delta$ for any $\delta > \gamma$, so $v(a-c) \notin \Gamma_K$.

Thus $K(a)$ is a ramified extension.

This is also a definable type.

types in ACVF

Examples: a is generic in an open ball

The type says that a is in the open ball $B^{op}_{\gamma}(c)$ and a is not in any smaller ball defined over K.
Then $v(a-c) > \gamma$, but $v(a-c) \not\geq \delta$ for any $\delta > \gamma$, so $v(a-c) \notin \Gamma_K$.
Thus $K(a)$ is a ramified extension.

This is also a definable type.

Examples: a is generic in an infinite intersection of balls

The type says that a is in $B_{\gamma_i}(c_i)$ for an infinite nested sequence of balls (and not generic in an open or closed ball).
Then $\Gamma_{K(a)} = \Gamma_K$ and $k_{K(a)} = k_K$, for otherwise a would be put into some K-definable open or closed ball. Thus $K(a)$ is an immediate extension.

This type is not definable.

imaginaries

An *imaginary* is an equivalence class of an $\emptyset$-definable equivalence relation E. The imaginary is *eliminated* if there is an $\emptyset$-definable function $f : K^n \to K^m$ such that $f(x) = f(y)$ if and only if xEy.

Examples of imaginaries in ACVF

$xEy \iff v(x) = v(y)$ In the sorted language, $v : K \to \Gamma$ is the function which eliminates the imaginary.

imaginaries

An *imaginary* is an equivalence class of an $\emptyset$-definable equivalence relation E. The imaginary is *eliminated* if there is an $\emptyset$-definable function $f : K^n \to K^m$ such that $f(x) = f(y)$ if and only if xEy.

Examples of imaginaries in ACVF

$xEy \iff v(x) = v(y)$ In the sorted language, $v : K \to \Gamma$ is the function which eliminates the imaginary.

$xEy \iff v(x) = v(y) = 0 \wedge v(x - y) > v(x)$ is eliminated by the function

$$f(x) = \begin{cases} x & \text{if } v(x) \neq 0; \\ \operatorname{res}(x) & \text{if } v(x) = 0. \end{cases}$$

in the sorted language.

imaginaries

An *imaginary* is an equivalence class of an $\emptyset$-definable equivalence relation E. The imaginary is *eliminated* if there is an $\emptyset$-definable function $f : K^n \to K^m$ such that $f(x) = f(y)$ if and only if xEy.

Examples of imaginaries in ACVF

$xEy \iff v(x) = v(y)$ In the sorted language, $v : K \to \Gamma$ is the function which eliminates the imaginary.

$xEy \iff v(x) = v(y) = 0 \wedge v(x-y) > v(x)$ is eliminated by the function

$$f(x) = \begin{cases} x & \text{if } v(x) \neq 0; \\ \operatorname{res}(x) & \text{if } v(x) = 0. \end{cases}$$

in the sorted language.

$xEy \iff v(x) = v(y) \wedge v(x-y) > v(x)$ should be coded by the pair $(v(x), B^{op}_{v(x)}(x))$, except that the ball is not in the language.

how to eliminate imaginaries

Add sorts to the language for the following sets:

$$S_n = \{s : s \text{ is a free } \mathcal{O}\text{-submodule of } K^n \text{ of rank } n\}$$

In particular,

$$S_1 = \{\mathcal{O}a : a \in K\} = \{B^{cl}_{v(a)}(0) : a \in K\}.$$

how to eliminate imaginaries

Add sorts to the language for the following sets:

$$S_n = \{s : s \text{ is a free } \mathcal{O}\text{-submodule of } K^n \text{ of rank } n\}$$

In particular,

$$S_1 = \{\mathcal{O}a : a \in K\} = \{B^{cl}_{v(a)}(0) : a \in K\}.$$

For any $s \in S_n$, $s/\mathfrak{m}s = \{a + \mathfrak{m}s : a \in s\}$ is an n-dimensional k-vector space. Let

$$T_n = \bigcup_{s \in S_n} s/\mathfrak{m}s.$$

Then

$$T_1 = \bigcup_{s \in S_1} \{a + \mathfrak{m}s : a \in s\} = \bigcup_{s \in S_1} \{B^{op}_{v(a)}(a) : a \in s\}.$$

Theorem

Let $\mathcal{G}$ be the collection of sorts $K, k, \Gamma, \mathcal{S} = \bigcup S_n, \mathcal{T} = \bigcup T_n$ and let $\mathcal{L}_{\mathcal{G}}$ be a language with appropriate functions on each sort. The theory ACVF has elimination of imaginaries in $\mathcal{L}_{\mathcal{G}}$.

stable part

stable theories by example

- An algebraically closed field is stable.
- A vector space over an algebraically closed field is stable.
- An ordered group is not stable.
- A valued field is not stable.

stable part

stable theories by example

- An algebraically closed field is stable.
- A vector space over an algebraically closed field is stable.
- An ordered group is not stable.
- A valued field is not stable.

stable, stably embedded sets

A definable set D in K is *stable* if the structure with universe D and relation symbols for $D \cap E$ for all definable sets E in K is stable.
A definable set D is *stably embedded* if for all definable sets E in K, $D \cap E$ is definable with parameters from D.
k is stably embedded (by quantifier elimination) and stable (because it is an algebraically closed field),
Γ is stably embedded by qe, but is not stable (because it is an ordered group)

stable part

Definition

Given parameters C, the *stable part over* C, St_C, is the multi-sorted structure whose sorts are the stable, stably embedded sets defined over C, with the structure induced from the ambient structure. For any set A, $\mathrm{St}_C(A) = \mathrm{dcl}(CA) \cap \mathrm{St}_C$. Notice that St_C is stable.

stable part

Definition

Given parameters C, the *stable part over* C, St_C, is the multi-sorted structure whose sorts are the stable, stably embedded sets defined over C, with the structure induced from the ambient structure. For any set A, $\mathrm{St}_C(A) = \mathrm{dcl}(CA) \cap \mathrm{St}_C$. Notice that St_C is stable.

Definition

Given parameters C in a model of ACVF, we denote by $\mathrm{VS}_{k,C}$ the multi-sorted structure with sorts $s/\mathfrak{m}s$, where $s \in S_n(C)$ (which are all stable, stably embedded).

Fact

In ACVF, up to interpretability, $\mathrm{VS}_{k,C}$ and St_C are the same structure. In particular, if $C \models ACVF$ then St_C is precisely k_C.

stably dominated types

A stable theory has a notion of independence: we write $A \downarrow_C B$ and say A is independent from B over C, if $\mathrm{tp}(A/C) \vdash \mathrm{tp}(A/B)$ (the type of A over B has no more information than the type of A over C).
Another way to say this (if the base C is algebraically closed) is: whenever σ is an automorphism fixing C pointwise and mapping A to A', there is an automorphism τ fixing B pointwise and mapping A to A'.

stably dominated types

A stable theory has a notion of independence: we write $A \downarrow_C B$ and say A is independent from B over C, if $\mathrm{tp}(A/C) \vdash \mathrm{tp}(A/B)$ (the type of A over B has no more information than the type of A over C).
Another way to say this (if the base C is algebraically closed) is: whenever σ is an automorphism fixing C pointwise and mapping A to A', there is an automorphism τ fixing B pointwise and mapping A to A'.

Definition

$\mathrm{tp}(A/C)$ is *stably dominated* if, whenever $C \subset B$ and $\mathrm{St}_C(A) \downarrow_{\mathrm{St}_C} \mathrm{St}_C(B)$ then

$$\mathrm{tp}(A/C\mathrm{St}_C(B)) \vdash \mathrm{tp}(A/CB);$$

that is, if there is an automorphism σ fixing $C \cup \mathrm{St}_C(B)$ pointwise and mapping A to A' then there is an automorphism τ fixing $C \cup B$ and mapping A to A'.

examples around stably dominated types in ACVF

$\mathrm{tp}(a/K)$ generic in a closed ball

Then $\mathrm{tp}(a/K)$ is stably dominated.
We already observed that $K(a)/K$ is a residual extension; $k_{K(a)} \neq k_K$.
Let L be another field. To say that $\mathrm{St}_K(a) \downarrow_{\mathrm{St}_K} \mathrm{St}_K(L)$ means precisely that $\mathrm{res}(a) \downarrow_k k_L$, or that $\mathrm{res}(a) \notin k_L$. Then any automorphism fixing k_L and mapping a to a' will extend to an automorphism fixing all of L and mapping a to a'.

examples around stably dominated types in ACVF

$\mathrm{tp}(a/K)$ generic in a closed ball

Then $\mathrm{tp}(a/K)$ is stably dominated.
We already observed that $K(a)/K$ is a residual extension; $k_{K(a)} \neq k_K$.
Let L be another field. To say that $\mathrm{St}_K(a) \downarrow_{\mathrm{St}_K} \mathrm{St}_K(L)$ means precisely that $\mathrm{res}(a) \downarrow_k k_L$, or that $\mathrm{res}(a) \notin k_L$. Then any automorphism fixing k_L and mapping a to a' will extend to an automorphism fixing all of L and mapping a to a'.

$\mathrm{tp}(a/K)$ generic in an open ball

Then $\mathrm{tp}(a/K)$ is not stably dominated.
In this case, $K(a)/K$ is a ramified extension.
Because $k_{K(a)} = k_K$, the independence condition $\mathrm{St}_K(a) \downarrow_{\mathrm{St}_K} \mathrm{St}_K(L)$ gives no information about the relationship between L and a.
Suppose $0 < v(a) < \gamma$ for all $\gamma \in \Gamma_K$, but there is $\delta \in L$ with $0 < \delta < v(a)$.
There can be an automorphism σ fixing k_L with $v(\sigma(a)) < \delta$, so no automorphism fixing L will map a to $\sigma(a)$.

orthogonality

Definition

$\mathrm{tp}(a/C)$ is *orthogonal to* Γ if for every model K extending C, $\Gamma_{Ka} = \Gamma_K$.

Theorem

$\mathrm{tp}(a/C)$ is stably dominated if and only if it is orthogonal to Γ.

Example

The type of an element a which is generic in an infinite intersection of balls over K is not stably dominated.
For let L be an extension of K which puts another element into the same intersection. Then $\Gamma_{La} \neq \Gamma_L$.

maximally complete base

Definition

K is *maximally complete* if it has no proper immediate extensions.

Theorem

Let $K \models ACVF$ be a maximally complete field, a a new element. Then $\mathrm{tp}(a/K \cup \Gamma_{K(a)})$ is stably dominated.

In field-theoretic terms: let L be an extension of K with $\mathrm{res}(a) \notin k_L$. Then if there is an isomorphism fixing k_L and $\Gamma_{K(a)}$ and mapping $K(a)$ to $K(a')$ then there is such an isomorphism fixing all of L.

maximally complete base

Definition

K is *maximally complete* if it has no proper immediate extensions.

Theorem

Let $K \models ACVF$ be a maximally complete field, a a new element. Then $\text{tp}(a/K \cup \Gamma_{K(a)})$ is stably dominated.

In field-theoretic terms: let L be an extension of K with $\text{res}(a) \notin k_L$. Then if there is an isomorphism fixing k_L and $\Gamma_{K(a)}$ and mapping $K(a)$ to $K(a')$ then there is such an isomorphism fixing all of L.

Martin Hils will talk about the connection between stably dominated types and berkovich space.