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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Alex Kruckman **Email/Phone:** kruckman@gmail.com

Speaker's Name: Antoine Chambert-Loir

Talk Title: Specialities for non-specialists (II)

Date: 02/04/14 **Time:** 9:30 am / pm (circle one)

List 6-12 key words for the talk: Special points, Manin-Mumford, Mordell-Lang, André-Oort, Zilber-Pink, unlikely intersections

Please summarize the lecture in 5 or fewer sentences: Part 2 of 3. A general framework for formulating propositions about "special points" and "special subspaces" was introduced, along with several general principles stated in this abstract language. The theorems/conjectures of Manin-Mumford, Mordell-Lang, and André-Oort as specific cases of this framework. At the end of the lecture, abelian varieties and the theorems of Mordell-Weil and Lang-Néron,

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☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
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- **Handouts**: Obtain copies of and scan all handouts

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☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

3. A few principles

Special?

Framework: V a topological space given with (some) closed subspaces called _special subspaces_. Special points are special points.

i.e. T_0: pts are distinguished by open sets — no infinite descending chain of closed sets — good notion of dimension (length of chains of closed subsets)

Assumptions (1) V is a Kolmogoroff noetherian catenary top. space

(2) special subspaces are closed under finite unions and arbitrary intersections $\rightsquigarrow$ special topology

(3) Every irreducible component of a special subspace is special and contains a dense set of special points.

When $X \subset V$, one writes S_X for its special closure, i.e. the smallest special subspace containing X.

$\mathrm{defect}(X) = \mathrm{codim}(X, S_X)$

$\boxed{\text{Manin-Mumford, Mordell-Lang, Full Mordell-Lang, Bogomolov}}$

Principle 1: Let X be a closed subspace of V.

(1) There exists a special subspace $S \subset X$ such that $X \setminus S$ contains no special point.

(2) There exists a largest special subspace $S \subset X$.

(1) $\rightarrow$ (2): $T \subset X$ special. $T \setminus (S \cap T)$ is open in T. If nonempty, it contains a special point — absurd. So $T = S \cap T$, i.e. $T \subset S$.

(2) $\rightarrow$ (1): If there is a special point $x \in X \setminus S$, $S \cup \{x\}$ would be special.

$\boxed{\text{André-Oort}}$

Principle 1': The closure of any set of special points is a special subspace.

Principles 1 and 1' are equivalent by a similar argument.

<u>Principle 2</u>: $X \subset V$ irreducible with special closure S_X.

Pink (2005)

(a) The intersection of X with the union of all special subspaces of dim < defect(X) is not dense in X

(a') The closure of a set of points of defects $\leq d$ is a subspace of defect $\leq d$.

$X, Y \subset V$ closed, W irreducible component of $X \cap Y$.
W is <u>anomalous</u> if $\dim(W) > 0$ and $\dim(W) > \dim(X) + \dim(Y) - \dim(V)$.

(if we do not require irreducibility, we can take a single special $T = \bigcup_{i=1}^{n} S_i$.

Zilber

(b) There exists a finite family $(S_1, \ldots, S_n)$ of special subspaces of V, $\neq V$, such that, for every special subspace S, every anomalous component of $X \cap S$ is contained in one of the S_i

Bombieri-Masser-Zannier

"unlikely intersections"

(c) $X^{ano} = X \setminus \bigcup$ (anomalous components of $X \cap$ special subspace of codim $\dim(X)$)
Then X^{ano} is open in X, and $X \setminus X^{ano}$ contains only finitely many points of defect < codim(X)

<u>Manin-Mumford</u> V is a "semiabelian variety"

<u>ex</u>. $V = (\mathbb{C}^\times)^n$

abelian variety $\mathbb{C}^g/L$

special points = torsion points

$V = (\mathbb{C}^\times)^n$: tuples $(x_1, \ldots, x_n)$ of roots of unity

$V = \mathbb{C}^g/L$: images of $L_{\mathbb{Q}}$ in V

speial subvarieties = (finite unions of irreducible components of) algebraic subgroups, torsion cosets

☆ in char 0:

<u>tori</u> Laurent 1984

<u>ab var</u> Raynaud 1983

Hindry

<u>Mordell-Lang</u> V = torus, ab. variety ~~or~~ semi ab. variety

$\Gamma \subset V$ a finitely generated group

points of Γ are the special points

irreducible special subvarieties = translates of connected subgroups by a point in Γ.

* in char. 0: Faltings 1991
Vojta

<u>Full Mordell-Lang</u> $\Gamma_{\mathbb{Q}} = \{x \in V \mid \exists n \geq 1,\ nx \in \Gamma\}$

division group of Γ.

special points = points in $\Gamma_{\mathbb{Q}}$ Note: $\Gamma_{\mathbb{Q}}$ includes all torsion points, so full Mordell-Lang implies Manin-Mumford

irreducible special subvarieties = translates of connected subgroups by a point in $\Gamma_{\mathbb{Q}}$.

* Faltings, McQuillan

<u>André-Oort</u> V = Shimura variety

<u>ex</u>: $V = (\mathbb{C})^n$, $\mathbb{C} = H/SL(2,\mathbb{Z})$

$$\mathbb{C} \xleftarrow{\ j\ } H \longrightarrow H/SL(2,\mathbb{Z})$$

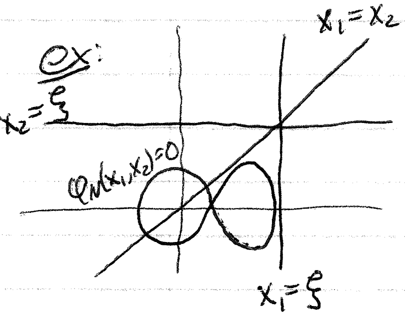
$(j_1, \ldots, j_n)$ <u>special</u> if there are elliptic curves with CM, $E_1, \ldots, E_n$ s.t. $j_K = j(E_K)\ \forall K$.

special subvarieties:

- $\{x_K = \xi\}$ $\xi = j(E)$, E has CM
- $\{x_K = x_e\}$
- Modular polynomials $\varphi_N \in \mathbb{Z}[X,Y]$ irreducible, $\varphi_N(j(z), j(Nz)) \equiv 0$

  $\{\varphi_N(x_K, x_e) = 0\}$

<u>ex</u>:



* Pila, 2012

<u>Ex</u> <u>Theorem</u> (Bombieri-Masser-Zannier)(Maurin)

$X \subset (\mathbb{C}^\times)^n$ a closed irreducible curve. Assume X is not contained in a proper subgroup. Then $\{x = (x_1, \ldots, x_n) \in X \mid x_1, \ldots, x_n$ satisfy 2 independent multiplicative dependence relations$\}$ is finite. ∎

Abelian varieties: Carrizosa, Galatean, Reimond-Viada. Habegger-Pila/$\overline{\mathbb{Q}}$.

Clarification: To satisfy 2 independent multiplicative dependence relations means $\exists (a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathbb{Z}^n$, non-proportional, s.t. $x_1^{a_1} \ldots x_n^{a_n} = x_1^{b_1} \ldots x_n^{b_n} = 1$.

4. Points in subvarieties of semiabelian varieties

k a field.

Abelian variety$/k$ = projective smooth connected algebraic group$/k$

automatically commutative

Thm (Mordell-Weil) If k is finitely generated over its prime field, then $A(k)$ (= group of k points) is finitely generated.

Thm (Lang-Néron) If k is finitely generated over k_0 ($k_0 \subset k$ regular), let A_0/k_0 be the k/k_0-trace of A. This is the "universal" abelian variety B/k_0 with a morphism $B_k \to A$.

$(A_0)_k \xrightarrow{\tau} A$, $A(k)/\tau(A_0(k_0))$ is finitely generated.