MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Alex Kruckman **Email/Phone:** kruckman@gmail.com

Speaker's Name: Pierre Simon

Talk Title: An Introduction to Stability-Theoretic Techniques (II)

Date: 02/04/14 **Time:** 11:00 am / pm (circle one)

List 6-12 key words for the talk: stable theories, definable types, forking, algebraic closure, strongly minimal sets

Please summarize the lecture in 5 or fewer sentences: Part 2 of 3. Slides with supporting boardwork. An introduction to model theoretic stability theory, with a focus on the notion of definable types. Nonforking independence and its properties, equivalent definitions of stability. Examples: defining types in the theory of algebraically closed fields, independence in separably closed fields. The talk concluded with a discussion of the acl pregeometry in strongly minimal sets.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

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- **Computer Presentations**: Obtain a copy of their presentation
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acl and dcl in algebraically closed fields

$A \subseteq K \models ACF$

$dcl(A) = \langle A \rangle^{-p^\infty}$, the perfect closure of the field generated by A

$acl(A) = \langle A \rangle^{acl}$, the algebraic closure of the field generated by A

Definable types

$p(\bar{x})$ a type over M, $\varphi(\bar{x};\bar{y})$ a formula

p is definable means $\{\bar{b} \in M^{|\bar{y}|} \mid \varphi(\bar{x};\bar{b}) \in p\}$ is a definable set for all $\varphi(\bar{x};\bar{y})$

$M = \overline{\mathbb{Q}}$, $p = tp(a_1, a_2/\overline{\mathbb{Q}})$, $a_1, a_2 \in \mathbb{C}$.

Case 1: $tr.deg(a_1, a_2/\overline{\mathbb{Q}}) = 0$, i.e. $a_1, a_2 \in \overline{\mathbb{Q}}$.

$\{\bar{b} \in \overline{\mathbb{Q}} \mid \varphi(\bar{x};\bar{b}) \in p\} = \{\bar{b} \in \overline{\mathbb{Q}} \mid \overline{\mathbb{Q}} \models \varphi(a_1, a_2, \bar{b})\}$

So $d\varphi(\bar{y}) = \varphi(a_1, a_2, \bar{y})$.

Case 2: $tr.deg(a_1, a_2/\overline{\mathbb{Q}}) = 2$.

$\varphi(x_1, x_2; \bar{y}): \sum_{i,j \le n} y_{ij} x_1^i x_2^j = 0$

$\sum_{i,j \le n} b_{ij} x_1^i x_2^j = 0 \in p \leftrightarrow \bigwedge_{i,j \le n} b_{i,j} = 0$, so $d\varphi(\bar{y}): \bigwedge_{i,j \le n} y_{ij} = 0$

Case 3: $tr.deg(a_1, a_2/\overline{\mathbb{Q}}) = 1$. $C(a_1, a_2) = 0$, $C(x_1, x_2) \in \overline{\mathbb{Q}}[x_1, x_2]$.

$\sum_{i,j \le n} b_{ij} x_1^i x_2^j = 0 \in p \leftrightarrow C(x_1, x_2) \mid \sum_{i,j \le n} b_{ij} x_1^i x_2^j$

$\sum_{i,j \le n} b_{ij} x_1^i x_2^j = \varphi(x_1, x_2; \bar{y})$

$C(x_1, x_2) \mid \sum_{i,j \le n} b_{ij} x_1^i x_2^j$ ⟶ $d\varphi(\bar{b}) / \overline{\mathbb{Q}}$

$(\mathbb{Q}, \le) \preceq (\mathbb{R}(t), \le)$ where $t > \mathbb{Q}$.

$M = (\mathbb{Q}, \le)$. Look at $p = tp(\sqrt{2}/M)$, $\varphi(x,y): x \le y$.

$\{b \in M \mid \varphi(x,b) \in p\} = \{b \in \mathbb{Q} \mid \sqrt{2} \le b\}$ NOT a definable set in $M = (\mathbb{Q}, \le)$.

$p(x) = tp(t/\mathbb{Q}) = +\infty$ is a definable type.

$\{b \in \mathbb{Q} \mid x \leq b \in p\} = \emptyset$, which is a definable set.

$q(x) = tp(1/t/\mathbb{Q}) = 0^+$ $(0 < 1/t < \mathbb{Q}_{>0})$ is a definable type.

$\{b \in \mathbb{Q} \mid x \leq b \in p\} = \{b \in \mathbb{Q} \mid b > 0\}$, a definable set.

Pushforwards

If f is a definable function and $p = tp(a/M)$,

$f_* p = tp(f(a)/M)$.

p definable $\Rightarrow$ $f_* p$ definable

Extending definable types

We have $\bar{a}$, $M \leq M\bar{b}$.

$p = tp(\bar{a}/M)$ definable: For any $\varphi(\bar{x}, \bar{y}) \rightsquigarrow d\varphi(\bar{y})$,

and $\varphi(\bar{x}, \bar{b}') \in p \Leftrightarrow \mathcal{U} \models d\varphi(\bar{b})$ for $\bar{b}' \in M$.

We define $p \mid M\bar{b}$ as the type q over $M\bar{b}$, defined by

$\varphi(\bar{x}, \bar{b}') \in q \Leftrightarrow \mathcal{U} \models d\varphi(\bar{b}')$ for $\bar{b}' \in M\bar{b}$.

This does give a type (small argument, uses that M is a model).

Question: Does the type $p \mid M\bar{b}$ depend on the definition scheme $d\varphi$?

Answer: No, since M is a model, any two definition schemes must be equivalent, according to the theory.

In ACF $(\mathbb{C})$, $\bar{a} \perp_K \bar{b}$ iff $tr.deg(\bar{a}/K) = tr.deg(\bar{a}/K\bar{b})$.

Multiple nonforking extensions

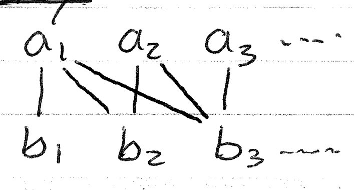
$A = \mathbb{Q} \subseteq \mathbb{C}$ (A is not a model)

$p = tp(\sqrt{2}/\mathbb{Q})$ over $\mathbb{C}$: $\nearrow x = \sqrt{2}$, $\searrow x = -\sqrt{2}$

p should have a nonforking extension, and there is no canonical choice. Both are nonforking.

The order property

Picture: φ



Example Separably closed fields

Def K is *separably closed* if it has no finite separable extensions, i.e. $K^{alg} = K^{-p^\infty}$.

$[K : K^p] = p^n$, n is the *degree of imperfection* of K

Find $e_1, \dots, e_n \in K$, a p-basis:

$\forall a \in K$, $$a = \sum_{0 \le i_1, \dots, i_n < p} (f_{i_1, \dots, i_n}(a))^p e_1^{i_1} \dots e_n^{i_n}.$$

(We are writing the coefficients in K^p explicitly as functions of a)

Expand the language to include $e_1, \dots, e_n$ constants, and $\{f_\sigma(x) \mid \sigma = (i_1, \dots, i_n) \in p^n\}$ functions. $\Rightarrow$ Quantifier elimination.

Then $a \underset{K}{\perp} b \iff a, f_\sigma(a), f_{\sigma'}(f_\sigma(a)), \dots \underset{K}{\perp}^{alg} b, f_\sigma(b), f_{\sigma'}(f_\sigma(b)), \dots$

The language of vector spaces

V a K-vector space is viewed as a structure $(V; +, 0, \{\lambda_r(x) \mid r \in K\})$

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acl in strongly minimal sets

Below $A \subseteq X$ strongly minimal.

$A \subseteq acl(A) = acl(acl(A))$ (it is a closure operator)

If $a \in acl(A \cup \{b\}) \setminus acl(A)$, then $b \in acl(A \cup \{a\})$ (exchange)

These properties give rise to a dimension function, just as in linear algebra.

Introduction to model theoretic techniques

Pierre Simon

Université Lyon 1, CNRS and MSRI

Introductory Workshop: Model Theory, Arithmetic Geometry and Number Theory

Let $A \subseteq M$.

- Definable closure $dcl(A)$:
  $e \in dcl(A)$ if there is $\varphi(x; \bar{a}) \in \mathrm{tp}(e/A)$ such that e is the only element in M satisfying $\varphi(x; \bar{a})$.

  Equivalently, $e \in dcl(A)$ if and only if $e = f(\bar{a})$ for some definable function f and tuple $\bar{a}$ of elements of A.
- Algebraic closure $acl(A)$:
  $e \in acl(A)$ if there is $\varphi(x; \bar{a}) \in \mathrm{tp}(e/A)$ such that there are finitely many elements in M satisfying $\varphi(x; \bar{a})$.

Definable types

Definition

A type $\text{tp}(\bar{a}/M)$ is *definable* if for every formula $\varphi(\bar{x};\bar{y})$, there is a formula $d\varphi(\bar{y})$ with parameters in M, such that for any tuple $\bar{b} \in M$:

$$M \models \varphi(\bar{a};\bar{b}) \iff M \models d\varphi(\bar{b})$$

Examples: ACF, $(\mathbb{Q}, \leq)$.
Pushforward $f_* p$.

Stable theories

Definition

A theory T is *stable* if all types over all models of T are definable.

Examples:

- ACF;
- abelian groups;
- DCF_0: differentially closed fields of char 0;
- $\mathrm{SCP}_{p,n}$: separably closed fields.

Some unstable theories:

- $Th(\mathbb{R}, 0, 1, +, -, *, \leq)$;
- valued fields.

Independence (non-forking)

Definition

(T is stable) We say that $\bar{a}$ is independent from $\bar{b}$ over M, or $\mathrm{tp}(\bar{a}/M\bar{b})$ *does not fork* over M, written

$$\bar{a} \underset{M}{\downarrow} \bar{b}$$

if $\mathrm{tp}(\bar{a}/M\bar{b})$ is according to the definition scheme of $\mathrm{tp}(\bar{a}/M)$.

Examples: ACF, divisible torsion free abelian groups.

In stable theories, we can generalize this definition to an arbitrary base set A instead of M.

Some properties of independence

- **Existence** Let $p \in S(A)$ and $A \subseteq B$, then there is $q \in S(B)$ extending p and non-forking over A.
- **Algebraic closure** $c \mathop{\smile\!\!\!\!\!\!|}_A c$ if and only if $c \in acl(A)$.
- **Transitivity** $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_A \bar{b}, \bar{c}$ iff $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_A \bar{b}$ and $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_{A,\bar{b}} \bar{c}$
- **Symmetry** $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_A \bar{b}$ iff $\bar{b} \mathop{\smile\!\!\!\!\!\!|}_A \bar{a}$
- **Uniqueness** if M is a model, $p \in S(M)$ and $M \subseteq B$, then p has a unique non-forking extension to a type over B.
  If we replace M by an aribtrary subset A, then p may have up to $2^{\aleph_0}$ non-forking extensions over B.

Stable formulas

Definition

A formula $\varphi(\bar{x}; \bar{y})$ has the order property if for every n, we can find tuples $\bar{a}_1, \ldots, \bar{a}_n$ and $\bar{b}_1, \ldots, \bar{b}_n$ such that:

$$\varphi(\bar{a}_i, \bar{b}_j) \iff i \leq j.$$

Fact

Let M be a structure (in a countable language), $T = Th(M)$ and $M \prec \mathcal{U}$ a monster model. The following are equivalent:

- *T is stable;*
- *no formula $\varphi(\bar{x}; \bar{y})$ has the order property;*
- *for any $\bar{a} \in \mathcal{U}$, $B \subset \mathcal{U}$, $\mathrm{tp}(\bar{a}/B)$ is definable;*
- *for any $B \subset \mathcal{U}$, there are at most $|B|^{\aleph_0}$ types over B.*

Example: Separably closed fields.

Geometric stability theory

Definition

A definable set X is *strongly minimal* if any definable subset of X is finite or cofinite.

Examples:

- An infinite set with no structure;
- A k-vector space V;
- An algebraically closed field.

If X is a strongly minimal set, the algebraic closure operator $acl(A)$ satisfies exchange and therefore gives rise to a dimension function $\dim(A)$ on subsets of X.

We classify such sets X according to the behavior of *acl*:

Disintegrated $acl(A) = \bigcup_{a \in A} acl(\{a\})$;

Locally modular $\dim(A \cup B) + \dim(A \cap B) = \dim(A) + \dim(B)$ for A, B closed, $\dim(A \cap B) \geq 1$;

Not locally modular The condition above does not hold.

Definition

A type $p(x)$ is minimal if for every formula $\varphi(x)$, either $p(\mathcal{U}) \cap \varphi(\mathcal{U})$ or $p(\mathcal{U}) \setminus \varphi(\mathcal{U})$ is finite.

If $p(x)$ is a minimal type, then as in the case of strongly minimal formulas, one considers the algebraic closure operator $acl(A)$ on subsets $A \subset p(\mathcal{U})$ and the associated dimension function.

End of talk 2.