MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Alex Kruckman **Email/Phone:** Kruckman@gmail.com

Speaker's Name: Ekaterina Amerik

Talk Title: Some Applications of Hrushovski's theorem about the Frobenius map to algebraic dynamics

Date: 02/04/14 **Time:** 1 : 30 am / pm (circle one)

List 6-12 key words for the talk: Algebraic dynamics, periodic points, Frobenius twist, Lang-Weil estimates

Please summarize the lecture in 5 or fewer sentences: The speaker demonstrated how Hrushovski's "twisted Lang-Weil estimates" can be used to show the existence of both periodic and non-preperiodic points for a dominant rational maps in the context of algebraic dynamics.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
  - Overhead: Obtain a copy or use the originals and scan them
  - Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - Handouts: Obtain copies of and scan all handouts
- ☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

<u>Th</u> (Hrushovski) X/k affine variety, k alg. closed, $\mathrm{char}(k)=p\neq 0$,
$\varphi_q: x\longmapsto x^q$, X^{φ_q} "Frobenius twist" of X.
$\Gamma_{\varphi_q}\subset X\times X^{\varphi_q}$. Let $S\subset X\times X^{\varphi_q}$, $\dim S=\dim X=d$,
projections dominant, one of them quasifinite.
Then $|S\cap\Gamma_{\varphi_q}| = aq^d + O(q^{d-\frac{1}{2}}) \longrightarrow \leq C_1 q^{d-\frac{1}{2}}$ when $q\geq C_2$

$a = [k(S):k(X)]/[k(S):k(X^{\varphi_q})]$ purely inseparable

C_1, C_2 depend only on certain discrete invariants of X, S.

<u>Cor</u> If $k=\overline{\mathbb{F}_p}$, (then for some q, $X^{\varphi_q}=X$), take $S\subset X^2$ satisfying the condition of the theorem, $|S\cap\Gamma_{\varphi_q^m}|\neq\emptyset$ for $m\gg 0$.

<u>Remark</u> Previous results of this type have stronger hypotheses on S: one projection proper, another quasi-finite. Even if S is the graph of a rational map, this mostly can't be achieved.

<u>Variant</u> $\forall W\subset X$ subvariety, $\exists m$ s.t. $\exists u\in X\setminus W$ with $(u,\varphi_q^m u)\in S$.

<u>Cor</u> (Fakhruddin, Poonen) $f: X\longrightarrow X$ (alg. variety) over $\mathbb{F}_p$ finite surjective.
Periodic points of f are Zariski dense.
<u>Pf</u>: Suppose [periodic pts] $\subset Y$ proper subvariety.
Take $\mathcal{U}\subset X\setminus Y$ affine. Apply Hrushovski to $\mathcal{U}\times\mathcal{U}$, $S=\Gamma_f|_{\mathcal{U}\times\mathcal{U}}$.
$\Rightarrow \exists u\in\mathcal{U}$ s.t. $f(u)=\varphi_q^m(u)$. But such u is φ_q-periodic,
$\Rightarrow$ also f-periodic.

<u>Cor</u> (A.) Same for $f: X \dashrightarrow X$ dominant rational.
"Difficulty": one can have $f^i(u)\in \mathrm{Indet}(f)$.
Have to consider u s.t. $f^i(u)\in\mathrm{Indet}(f)$ for <u>any</u> i.
$\Rightarrow$ <u>remove</u> $f^{-i}(\mathrm{Indet}(f))$. <u>A priori</u>, $f^{-i}(\mathrm{Indet}(f))$ can exhaust $X(\overline{\mathbb{F}_p})$. <u>But</u>: Let $W=\bigcup_{i\in\mathbb{Z}}\varphi_q^i(\mathrm{Indet}(f))$. This is a <u>finite</u> union.
Take $u\notin W$ s.t. $f(u)=\varphi_q^m(u)$. This is <u>really</u> periodic!
$\rightarrow$ Moreover, can remove other subvarieties to get periodic pts with special properties. For instance, $V=\bigcup\varphi_q^i(\mathrm{Ram}(f))$, $Df=0$
$\Rightarrow$ get p.p. never landing in $\mathrm{Ram}(f)$.

<u>Fakhruddin's Thm</u> X proj. var$/k$, $k=\bar{k}$, $f: X \to X$ dominant morphism, L line bundle (ample), s.t. $f^*L \otimes L^{-1}$ is ample. (<u>Note</u>: f finite).

Then periodic points are Zariski dense.

<u>Pf</u>: (for $k=\bar{\mathbb{Q}}$, the case of $\mathrm{tr.deg}_{\bar{\mathbb{Q}}}\, k = n$ or $\mathrm{tr.deg}_{\bar{\mathbb{F}}_p}\, k = n$ is obtained by same reductions)

Meaning of $f^*L \otimes L^{-1}$ ample: $(*)$

$[\text{Fixed pts of } f] = Y$ subvariety

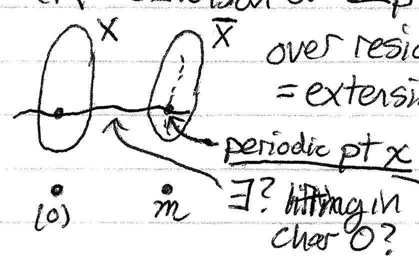
$(*)$ means $\dim Y = 0$, <u>since</u> on Y, $f^*L \otimes L^{-1}$ is trivial and ample at the same time. $\dim[\text{Fixed pts of } f^n]$ is also 0

$\quad\quad$ $\text{pts} = $ per. pts. of period dividing n.

$(*) \Rightarrow (f^n)^* L \otimes L^{-1}$ ample.

Choose a model $\mathcal{X}$ over $\mathrm{Spec}\, R$, R a DVR.

($R=$ extension of $\mathbb{Z}_p$ for well-chosen p)



over residue field $=$ extension of $\mathbb{F}_p$

$f \rightsquigarrow \mathcal{F}: \mathcal{X} \to \mathcal{X}$

$\bar{F}: \bar{X} \to \bar{X}$

intersection of $\Gamma_{\mathcal{F}^n}$ and Δ (diagonal)

here, many per. points for instance x, smooth on $\bar{X}$

Let $n=\dim X$. $\dim \mathcal{X} = n+1$. $\dim \mathcal{X} \times_{\mathrm{Spec} R} \mathcal{X} = 2n+1$.

$\dim \Delta_{\mathcal{X}} = n+1$, $\dim \Gamma_{\mathcal{F}^n} = n+1$.

$\Rightarrow \Delta_{\mathcal{X}} \cap \Gamma_{\mathcal{F}^n}$ is at least 1-dimensional.

<u>so</u>, either x lifts to X, or $\exists$ curve of per. pts. on $\bar{X}$.

The second option contradicts $(*)$.

<u>Thm</u> (A.) $f: X \dashrightarrow X/\bar{\mathbb{Q}}$ dominant rational of infinite order.

Then $\exists$ points$/\bar{\mathbb{Q}}$ with infinite orbit (non-preperiodic) (dense in analytic topology).

<u>Issue</u>: Countability of $\bar{\mathbb{Q}}$! If we replace $\bar{\mathbb{Q}}$ by an uncountable field, this is obvious.

<u>But</u> even the existence of points$/\bar{\mathbb{Q}}$ with well-defined orbit follows from Hrushovski.

Why do we want to know such things?

Reasons: ① If we don't understand these, we don't understand anything.

② $\exists$ conjectures about "uniform behaviour of orbits"

Zhang, A.-Campana, Medvedev-Scanlon

$f: X \dashrightarrow X/\mathbb{C}$, $\exists$ maximal Zar.
$$\begin{tikzcd} X_n \arrow[rr, dashed, "f^K"] \arrow[dr, dashed, "\pi"'] & & X \arrow[dl, dashed, "\pi"] \\ & T_s & \end{tikzcd}$$

$\Rightarrow$ orbits of general pt. $/\mathbb{C}$ have dim. $d-s$.

Conj: $\exists\, x \in X(\overline{\mathbb{Q}})$ with orbit Z dense in a variety of dim $d-s$.

Thm: $s \neq d \Rightarrow$ Zar. closure of orbit of $x \in X(\overline{\mathbb{Q}})$ of dim > 0.

Pf: Replace f by a power $\Rightarrow$ for a suitable p, we have, by Hrushovski, a fixed pt. u mod p. On $X(K_p)$, K_p a suitable extension of $\mathbb{Q}_p$, we have an invariant p-adic nbhd $\mathcal{U}_p$ = everything reducing to u mod p. $\mathcal{U}_p = \mathfrak{p}^{\times n}$ ($\mathfrak{p}$ max ideal in $\mathcal{O}_p$, Frac $\mathcal{O}_p = K_p$),

f given by a convergent power series in $\mathcal{O}_p[[x_1, \dots, x_n]]$ with constant term divisible by the uniformizer π ($f(0) \in \mathfrak{p}$).

Change variables: $F(x) = \frac{1}{\pi} f(\pi x)$. $\mathcal{U}_p$ identifies with $\mathcal{O}_p^{\times n}$,

F linear mod $\mathfrak{p}$: $F = A + BX$, $F^i(u)$ never in the Ram $\Rightarrow B$ nondegenerate.

Replacing F by further iterate, suppose Id mod p.

Thm (Bell-Ghioca-Tucker, Poonen) $\forall x \in \mathcal{O}_p^{\times n}$ $\exists G: \mathbb{Z}_p \to \mathcal{O}_p^{\times n}$ analytic s.t. $G(i) = F^i(x)$ $\forall i \in \mathbb{Z}$.

x preperiodic $\Rightarrow G$ takes finitely many values $\Rightarrow G$ = constant.

Conclusion in $\mathcal{U}_p$, preperiodic points are periodic of bounded period

$\Rightarrow$ in a proper subvariety!

$\Rightarrow \exists$ pts $/\overline{\mathbb{Q}}$ which are not preperiodic.