MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Alex Kruckman **Email/Phone:** Kruckman@gmail.com

Speaker's Name: Martin Hils

Talk Title: A Model Theoretic Approach to Berkovich Spaces (I)

Date: 02/04/14 **Time:** 2:30 am /(pm)(circle one)

List 6-12 key words for the talk: ACVF, Berkovich spaces, definable types, orthogonality

Part 1 of 3

Please summarize the lecture in 5 or fewer sentences: The talk began with a motivation of Berkovich spaces. Berkovich's definition was given, and the structure of the Berkovich affine line was sketched. We then turned to the goals of the tutorial: introducing the model theoretic approach of Hrushovski and Loeser. After a brief review of the model theory of ACVF, the speaker explained what is meant by a definable type orthogonal to the value group

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
- **Overhead**: Obtain a copy or use the originals and scan them
- **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
- **Handouts**: Obtain copies of and scan all handouts

☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

<u>Notation:</u>

$$\begin{array}{lll} K & \xrightarrow{\text{val}} & \Gamma \cup \{\infty\} \\ \cup\!\!| & & \\ \mathcal{O}_K & \xrightarrow{\text{res}} & k_K \\ \cup\!\!| & & \\ m_K & & \end{array}$$

$B_{\geq\gamma}(a)$ closed ball } of radius γ
$B_{>\gamma}(a)$ open ball } around a

The open balls define a topology on K, called the valuation topology, so that K is a topological field.

<u>Problem:</u> Both "open" and "closed" balls are open.
e.g. $\mathcal{O} = \bigcup_{x\in\mathcal{O}} x + m$, in particular $\mathcal{O}$ has no boundary.

Let V be an alg. variety /K. Then V(K) may be given a topology
- on $\mathbb{A}^n(K) = K^n$, take the product topology
- glue

V(K) is also totally disconnected.

If $\Gamma_K \leq (\mathbb{R}, +, <)$, then we may define $|\cdot| : K \to \mathbb{R}_{\geq 0}$, $|x| = e^{-\text{val}(x)}$.
$(K, |\cdot|)$ is a field with $|\cdot|$ a non-archimedean ~~norm~~ absolute value.
Any non-archimedean absolute value comes from a valuation.

<u>Exs:</u> $\mathbb{Q}_p$, $\mathbb{Q}_p^{alg}$, $\widehat{\mathbb{Q}_p^{alg}}$, $k((t))$, k with trivial abs. value
not complete

<u>Problem:</u> One wants to mimic what is done over $\mathbb{C}$ (or $\mathbb{R}$), using the topology from $|\cdot|$,
- to do analytic geometry /K
- using the topology in algebraic geometric situations.

<u>Berkovich's approach:</u> Add new points to V(K), to get a "nice" topological space. The spaces are called <u>Berkovich (analytic) spaces</u>.
We will sketch V_K^{an}, where V is alg. var. /K complete.

Let V be affine, $A = K[V]$.

<u>Def</u>: A <u>multiplicative seminorm</u> on A/K is a map $|\cdot|_v : A \to \mathbb{R}_{\geq 0}$ s.t.

- $|\cdot|_v$ coincides with $|\cdot|$ on K
- $|ab|_v = |a|_v |b|_v$
- $|a+b|_v \leq \max(|a|_v, |b|_v)$

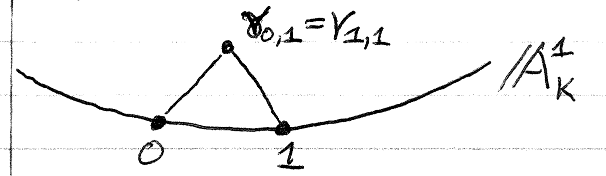
<u>Lemma</u>: (a) Let $(L, |\cdot|_L) \supseteq (K, |\cdot|_K)$ and $\bar{a} \in V(L)$.
Then $|F|_{\bar{a}} = |F(\bar{a})|_L$ is a multiplicative seminorm.
(b) Any multiplicative seminorm $|\cdot|_v$ is of this form, and $|\cdot|_{\bar{a}} = |\cdot|_{\bar{a}'}$ iff $(K(\bar{a}), |\cdot|) \simeq (K(\bar{a}'), |\cdot|)$.

As a set, $V_K^{an} = \{\text{mult. seminorms on } K[V]\}$.
Any $F \in K[V]$ induces $|F(-)|$ on V_K^{an}, $|F(\gamma)| = |F|_\gamma$.
On V_K^{an}, we take the coarsest topology s.t. every $|F| : V_K^{an} \to \mathbb{R}_{\geq 0}$ is continuous. As such, $V_K^{an} \subseteq^{cl} \mathbb{R}_{\geq 0}^{K[V]}$ with the subspace topology.

<u>Fact</u>: (1) V_K^{an} is locally compact and locally pathwise connected.
(2) In case $K \models ACVF$, then $V(K) \underset{dense}{\subseteq} V_K^{an}$, and $V(K)$ gets the subspace topology.
(The embedding is $\bar{a} \longmapsto |\cdot|_{\bar{a}}$)

<u>Ex</u>: $V = \mathbb{A}^1$, $K \models ACVF$, e.g. $K = \mathbb{C}_p$.
Let $r \geq 0$, define $|\sum c_i X^i|_{\gamma_{0,r}} = \max_i(|c_i| r^i)$
For $r = 0$, this corresponds to $0 \in \mathbb{A}^1(K)$.
For $r = 1$, this is the Gauss norm.
<u>Fact</u>: The map $i : \mathbb{R}_{\geq 0} \hookrightarrow \mathbb{A}_K^{1,an}$ is a homeomorphism onto its image. $r \longmapsto |\cdot|_{0,r}$
Similarly, for $a \in K$ arbitrary, define $|\sum_i c_i (x-a)^i|_{\gamma_{a,r}} = \max_i(|c_i| r^i)$.

$\mathbb{A}_K^{1,an}$ is treelike



<u>Note</u>: $|\cdot|_{\gamma_{a,r}} = |\cdot|_{\gamma_{b,s}}$ iff $r = s$ and $|a-b| \leq r$.
One even gets: $\mathbb{A}_K^{1,an}$ is contractible.

4 Kinds of points:

(1) Simple points (i.e. points in $\mathbb{A}^1(K)$)

(2) $|\cdot|_{a,r}$ with $0 < r \in |K|$, generic of a closed K-def ball
(≙ residual extension)

(3) $|\cdot|_{a,r}$ with $r \notin |K|$ (≙ ramified extension)

(4) other pts. (≙ immediate extension)

Some topological tameness properties

Berkovich
* V^{an} is locally compact and locally pathwise connected
* If V is smooth, V^{an} is locally contractible
* (under some hypotheses) V^{an} allows for a strong deformation retraction onto a finite simplicial complex

Ducros
* Every "semialgebraic" subset of V^{an} has finitely many connected components, all "semialgebraic".

Goal of the tutorial: Introduce some of the main ideas of Hrushovski and Loeser in recent work on a model-theoretic analog of V^{an}.
* They show very general topological properties for V^{an} (V quasiprojective)
* New foundations: Entirely new methods in non-archimedean (algebraic) geo.

Tools: Model theory of ACVF (Haskell-Hrushovski-Macpherson)

3 sources of topological tameness may be identified:
- definability ⇒ applications of compactness for uniform bounds
- stability (in the residue field, ACF)
- o-minimality ($(\Gamma, 0, +, <, \infty)$ is o-minimal, even piecewise linear)

Recall: ACVF in $\mathcal{L}_{k,\Gamma}$ has QE.

Cor: (1) any definable subset $X \subseteq K^n$ is semialgebraic, i.e. given by a boolean combination of conditions $F(\bar{x}) = 0$ or $val(F(\bar{x})) \leq val(G(\bar{x}))$, $F, G \in K[\bar{x}]$.

(2) k is a pure ACF

(3) $\Gamma_\infty^n \supseteq D$ def. is def. in $(\Gamma_\infty; 0, +, <, \infty)$.

(4) Any def. map $f: K^n \to \Gamma_\infty$ is, on a finite partition into definable sets, of the form $1/m\,(\mathrm{val}(F(\bar{x})) - \mathrm{val}(G(\bar{x}))$, where $m \geq 1$ and $F, G \in K[\bar{X}]$.

<u>Some notation</u>: Work in $\mathcal{G}$ (geometric sorts: $K, \Gamma_\infty, k, S_n, T_n$)

- Let X be a C-def. set, and let $C \subseteq A$. We denote by $S_{X,def}(A)$ the set of A-definable types on X (i.e. $p(x)$ is a definable type implying $x \in X$, and $d_p\varphi$ are formulas with parameters from A)
- $p \in S_X(A)$ is said to be <u>definable</u> if it extends uniquely to an A-def type over $\mathrm{dcl}(K) \supseteq A$.

<u>Def</u>: $p \in S_{X,def}(A)$. We say $p \perp \Gamma$ ("orthogonal") if for every $A \subseteq L \models ACVF$, $a \models p|L$, then for any L-def function $f: X \to \Gamma_\infty$, we have $f(a) \in \Gamma_\infty(L)$.

<u>Note</u>: These are precisely the def. types on X s.t. definable functions $f: X \to \Gamma_\infty$ extend to $f(p)$.
We may put $f(p) =$ the γ s.t. if $a \models p|L$, then $f(a) = \gamma$.
In otherwords, $p \perp \Gamma$ iff for any def $f: X \to \Gamma_\infty$, $f_* p$ is a constant type on Γ_∞ (i.e. contains $z = \gamma$).

<u>Exs</u> Let $\hat{X}(A)$ be the A-def types on X orthogonal to Γ.

(0) $X(A) = \mathrm{dcl}(A) \cap X \hookrightarrow \hat{X}(A)$ naturally

(1) Let B be a K-def. ball (open or closed), p_B the generic type of B. Swiss cheese decomposition $\Rightarrow p_B$ is a def. type $\in S_{\mathbb{A}^1, def}$

 * $p_{\mathcal{O}} \perp \Gamma$. Indeed, enough to check $f = \mathrm{val}(\sum_i c_i x^i) \in L[x]$.
 But if $a \models p_{\mathcal{O}}|L$, then $f(a) = \min_i(\mathrm{val}(c_i)) = \gamma \in \Gamma_\infty(L)$.

 * $p_{\mathfrak{m}} \not\perp \Gamma$. Indeed, $\mathrm{val}_* p_{\mathfrak{m}} = 0^+$, as if $a \models p_{\mathfrak{m}}|L$, then $0 < \mathrm{val}(a) < \Gamma(L)_{>0}$

<u>Cor</u>: $\hat{\mathbb{A}}^1 =$ the set of closed balls (radius ∞ included)