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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Alex Kruckman **Email/Phone:** Kruckman@gmail.com

Speaker's Name: Lou van den Dries

Talk Title: Model Theory and Multiplicative Combinatorics (I)

Date: 02/04/14 **Time:** 4:00 am / pm (circle one)

List 6-12 key words for the talk: multiplicative combinatorics, approximate subgroups, Hrushovski, Breuillard-Green-Tao, Lie groups

Please summarize the lecture in 5 or fewer sentences: Part 1 of 3. K-approximate subgroups. "Multiplicative combinatorics" as a noncommutative analogue of additive combinatorics. The role of logical limits, leading to approximations by Lie groups, the theorem of Breuillard, Green, and Tao, and a proof (sketch) of a simplified version. More in-depth notes can be found at http://www.math.uiuc.edu/~vddries/Bourbaki.pdf

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- [x] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [x] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
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Approximate Groups

[after Hrushovski and Breuillard, Green, Tao]

1° Multiplicative Combinatorics

2° Limits of approximate groups

3° Structure of approximate groups: BGT-theorem

4° Sketch of proof of BGT-theorem

5° Two applications (Next time)

Hrushovski: Stable group theory and approximate subgroups, J. AMS (2012)

BGT: The structure of approximate groups IHES Publ. (2012)

1° Multiplicative Combinatorics

Additive combinatorics is about the structure of subsets of abelian groups: Cauchy (1813), Schur (1916), Van der Waerden, Schnirelman, Mann, Kneser, Erdös, Freiman, Szemeredi, Rutsa, Bourgain, Gowers, Green, Tao, ...

Reference book: "Additive Combinatorics" by Tao & Van Vu

Multiplicative combinatorics is its extension to arbitrary groups. It is fair to say that Hrushovski introduced a new idea into this subject: modeling by Lie groups, and that BGT brought it to fruition. This is what these talks are about.

Conventions and notations

Throughout G is an ambient group, $X, Y \subseteq G$.

$XY := \{xy : x \in X, y \in Y\}$, $X^2 = XX$, $X^3 = XXX$, etc.

$X^{-1} := \{x^{-1} : x \in X\}$. We call X symmetric if $1 \in X$ and $X = X^{-1}$

$\langle X \rangle :=$ subgroup of G generated by X
$= \bigcup_n X^n$ if X symmetric

K, L : real numbers $\geqslant 1$

X is said to be a K-approximate group (in G) if X is symmetric and $X^2 \subseteq EX$ for some finite $E \subseteq G$ of cardinality $|E| \leq K$

We think of K as small and fixed and are interested in the structure of finite K-approximate groups X for $|X| \gg K$.

Some basic results:

X a K-appr. group $\Longrightarrow X^n$ a K^n-appr. group (easy)

X finite K-appr. group $\Longrightarrow |X^2| \leq K|X|$ (trivial)

For weak converses, the following is very useful:

<u>Rutsa's Covering Lemma</u> If $X, Y \subseteq G$ are finite and $|XY| \leq K|Y|$, then $X \subseteq EYY^{-1}$ with $|E| \leq K$.

Pf: Let $E \subseteq X$ be such that $eY \cap e'Y = \emptyset$ for all distinct $e, e' \in E$. Then $|E| \leq K$, and by taking E maximal we get $X \subseteq EYY^{-1}$. $\boxtimes$

<u>Cor</u>
$$\left.\begin{array}{l} X \text{ finite symmetric} \\ |X^5| \leq K|X| \end{array}\right\} \Longrightarrow X^2 \text{ is a } K\text{-appr. group.}$$

Pf: Apply Rutsa's Covering Lemma to $|X^4 X| \leq K|X|$.

Suppose $X \subseteq G$ is finite, symmetric, and $|X^2| \leq K|X|$.

If G is abelian, then X^2 is a K^5-approximate group, but for arbitrary G there is no L depending only on K such that X^2 is necessarily an L-approximate group.

Nevertheless:

Th (Tao; new recent proof by Petridis)

There exists a $64K^{12}$-approximate group $Y \subseteq X^4$ such that $X \subseteq EY$, $|E| \leq 4K^4$.

Slicing Lemma (Helfgott)

If $X \subseteq G$ is a K-approximate group and $H \leq G$, then $X^2 \cap H$ is a K^3-approximate group.

2° Limits of approximate groups

Fix K. To study the structure of finite K-approximate groups $X \subseteq G$ as $|X| \to \infty$, we consider their (logical) limits, for example, ultraproducts. (For details, see [BGT] or §4 in my Bourbaki seminar paper.)

Such a limit $X \subseteq G$ is still a K-approximate group, but in general not finite any more, instead it is pseudofinite and the role of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ is taken over by $\mathbb{N}^* \supseteq \mathbb{N}$, $\mathbb{Z}^* \supseteq \mathbb{Z}$, $\mathbb{R}^* \supseteq \mathbb{R}$, with $|X| \in \mathbb{N}^*$. We do have a "counting" measure μ on $\langle X \rangle$ normalized such that $\mu(X) = 1$, with $\mu(Y) \in \mathbb{R}$ for definable sets $Y \subseteq \langle X \rangle$.

Hrushovski: the properties of this measure yield a group morphism $\pi : \langle X \rangle \longrightarrow \mathcal{G}$ onto a locally compact group $\mathcal{G}$ such that $X^4 \supseteq \pi^{-1} \mathcal{U}$ for some neighborhood $\mathcal{U}$ of the identity in $\mathcal{G}$.

Yamabe's theorem on approximating locally compact groups by Lie groups allows modifying $\pi : \langle X \rangle \to \mathcal{G}$ to a group morphism $\rho : \langle Y \rangle \longrightarrow \mathcal{H}$ onto a connected Lie group $\mathcal{H}$ with definable K^6-approximate group $Y \subseteq X^4$ such that $Y \supseteq \rho^{-1} \mathcal{U}$ for some neighborhood $\mathcal{U}$ of the identity in $\mathcal{H}$, and $X \subseteq EY$ for some finite E.

Application to finite approximate groups of bounded exponent

We say a set $X \subseteq G$ has exponent $e \in \mathbb{N}^{\geq 1}$ if $x^e = 1$ for all $x \in X$.

Cor (Hrushovski) If $X \subseteq G$ is a finite K-approximate group and X^2 has exponent e, then X^4 contains a (finite) subgroup F of G such that

$$X \subseteq EF \quad \text{with} \quad |E| \leq L = L(K,e)$$

Sketch of Proof: going to the logical limit we have our nice group morphism $\rho : \langle Y \rangle \to \mathcal{H}$, and the exponent e assumption yields a neighborhood of the identity in the connected Lie group $\mathcal{H}$ with exponent e. $\therefore$ $\mathcal{H}$ is trivial.

As $\ker(\rho) \subseteq Y$, this gives $Y = \langle Y \rangle \subseteq X^4$, and the desired result now follows on general logical grounds. $\boxtimes$

3° BGT - theorem

If $X \subseteq G$ is a finite K-approximate group, then there is a K^6-approximate group $Y \subseteq X^4$ such that

(i) $X \subseteq EY$ with $|E| \leq L = L(K)$

(ii) $Y^{m(K)}$ contains a finite normal subgroup F of $\langle Y \rangle$ such that $\langle Y \rangle / F$ is d-nilpotent, $d \leq 3 \log_2 K$

A group H is said to be <u>d-nilpotent</u> $(d \in \mathbb{N})$ if $H = \langle u_1, \dots, u_d \rangle$ with $[u_i, u_j] \in \langle u_1, \dots, u_{j-1} \rangle$ whenever $1 \leq i < j \leq d$

Thus finite K-approximate groups are controlled in some sense by <u>nilpotent</u> groups, a reduction of sorts to the abelian case. A stronger version of the BGT-theorem gives even tighter control by so-called <u>coset nilprogressions</u>, which generalize symmetric arithmetic progressions in $\mathbb{Z}$.

We won't go into that here, but this more detailed result gives a (qualitative) generalization of earlier "inverse" theorems by Freiman and Ruzsa (and others) in the abelian case.

4° Sketch of proof of BGT-theorem

We consider a pseudofinite K-approximate group $X \subseteq G$. Following Hrushovski we get a definable K^6-approximate group $Y \subseteq X^4$ and a group morphism $\rho : \langle Y \rangle \to \mathcal{H}$ onto a connected Lie group $\mathcal{H}$, with good properties such as: $Y \supseteq \rho^{-1} \mathcal{U}$ for some neighborhood $\mathcal{U}$ of the identity in $\mathcal{H}$.

$H = \langle Y \rangle^{*} :=$ smallest definable subgroup of G containing Y

$= \{ y_1 \cdots y_r : r \in \mathbb{N}^*,\ y_1, \dots, y_r \text{ is a definable sequence in } Y \}$

$H \supseteq \langle Y \rangle$

By induction on $d := \dim \mathcal{H}$ we shall construct definable normal subgroups

$$\{1\} = H_0 \subseteq H_1 \leq \cdots \subseteq H_{2d+1} = H \quad \text{of } H$$

such that for <u>even</u> i : $H_{i+1} \subseteq Y H_i$ (so H_{i+1}/H_i is pseudofinite)

and for <u>odd</u> i : $H_{i+1} = u_i^{\mathbb{Z}^*} H_i$, $u_i \in Y$, and the pseudocyclic group H_{i+1}/H_i is central in H/H_i.

On general logical grounds plus some group theory this yields the BGT-theorem, but with a bound depending only on K instead of $3/\log_2 K$. The latter requires an additional step.

To prepare for this induction on $d = \dim \mathcal{H}$ we first use the "no small subgroups" property of Lie groups to shrink Y without changing $\langle Y \rangle$ or H, so that $\rho(Y^2)$ contains no nontrivial subgroup of $\mathcal{H}$.

For $g \in G$ we define its exit norm $|g| \in \mathbb{R}^*$ by

$$|g| = \begin{cases} 0 & \text{if } g^r \in Y \text{ for all } r \in \mathbb{N}^* \\ 1/r & \text{if } r \in \mathbb{N}^* \text{ is minimal with } g^r \notin Y \end{cases}$$

So $0 \le |g| \le 1$, and $|g| < 1 \iff g \in Y$

A <u>key</u> step in the proof is adapting Gleason's work on Hilbert's 5^{th} problem to get $C \in \mathbb{N}$ such that for all $g, h \in Y$:

$$|gh| \le C(|g| + |h|), \quad |ghg^{-1}| \le C|h|, \quad |[g,h]| \le C|g|\,|h|$$

$\therefore$ get definable $H_1 \trianglelefteq H$,

$$H_1 := \{h \in H : |h| = 0\} = \{h \in H : h^r \in Y \text{ for all } r \in \mathbb{N}^*\}$$

$H_1 \subseteq \ker(\rho) \subseteq Y$. If $d = 0$, then $\mathcal{H}$ is trivial, so $H_1 = Y = H$, and we are done.

Let $d > 0$. We can replace H and Y by H/H_1 and Y/H_1 without changing $\mathcal{H}$ and arrange in this way that $|h| > 0$ for all $h \neq 1$ in H.

As Y is pseudofinite we can take $u \in Y$ such that $|u| > 0$ is minimal: this is the main use of finiteness. Then $|u|$ is infinitesimal, which in view of $|[g,h]| \leq C|g||h|$ gives $u \in Z(H)$.

$\therefore \quad H_2 := u^{\mathbb{Z}^*} = \{u^r : r \in \mathbb{Z}^*\}$ is central in H.

We can now pass from $H, Y, \mathcal{H}$ to H/H_2, Y/H_2, $\mathcal{H}/\mathcal{H}_2$ where $\mathcal{H}_2 :=$ closure of $\rho(H_2 \cap \langle Y \rangle)$ in $\mathcal{H}$, so $\mathcal{H}_2$ is central in $\mathcal{H}$ and infinite, so $\dim \mathcal{H}/\mathcal{H}_2 < \dim \mathcal{H}$. This decrease in dimension gives what we want by induction.

End of Sketch

Comments [BGT] gets sharper results by working throughout in the setting of (ambient) local groups where not all products xy might exist, and using an extension of the Gleason-Yamabe results to this setting by Goldbring.

$3\log_2 K$ -bound: the key point is that the dimension of the connected nilpotent group $\mathcal{H}$ modulo its largest compact subgroup is $\le 3\log_2 K$