MSRI

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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Alex Kruckman **Email/Phone:** Kruckman@gmail.com

Speaker's Name: Antoine Chambert-Loir

Talk Title: Specialities for non-specialists (III)

Date: 02 / 05 / 14 **Time:** 9 : 30 am / pm (circle one)

List 6-12 key words for the talk: Mordell-Lang, Shimura varieties, André-Oort, special points

Please summarize the lecture in 5 or fewer sentences: Part 3 of 3. The first half of the talk concentrated on the Mordell-Lang conjecture, its variants, and its history. In the second half, Shimura varieties were defined (from scratch!), and the speaker explained how the André-Oort conjecture fits into the framework of special points and special subvarieties.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
- **Overhead**: Obtain a copy or use the originals and scan them
- **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
- **Handouts**: Obtain copies of and scan all handouts

☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

<u>Mordell-Weil</u> A ab. $/k$ number field $\Rightarrow A(k)$ f.g.

<u>Lang-Nerón</u> A ab. $/k$, k f.g. $/k_0 \Rightarrow A(k)/\tau(A_0(k_0))$ f.g.

If k is f.g. over its prime field $\Rightarrow A(k)$ f.g.

<u>Mordell - Lang conjecture</u>
1922 1965

$$1 \longrightarrow \underbrace{T} \longrightarrow \overbrace{G}^{\text{semiabelian variety}} \longrightarrow \underbrace{A}_{\text{ab. variety}} \longrightarrow 0$$

<u>torus</u>: $T_{k^{alg}} \simeq (\mathbb{G}_{m,k^{alg}})^n$

A semiabelian variety / alg. closed field k.
$\Gamma \subset A(k)$ f.g. group. division group of Γ: $\Gamma_{\mathbb{Q}} = \{x \in A(k) \mid \exists n \geq 1, nx \in \Gamma\}$
p prime number: $\Gamma_{p'} = \{x \in A(k) \mid \exists n \geq 1$ <u>prime to p</u>, $nx \in \Gamma\}$

<u>Thm</u> Assume $\mathrm{char}(k) = 0$. Let $V \subset A$ be a closed subvariety, $\Gamma \subset A(k)$ f.g. subgroup. There exists a finite family $B_1, \ldots, B_n$ of translates of semiabelian subvarieties by points of $\Gamma_{\mathbb{Q}}$ s.t.
$B_i \subset V$ for all i, $V(k) \cap \Gamma_{\mathbb{Q}} \subset \bigcup_{i=1}^{n} B_i(k) \cap \Gamma_{\mathbb{Q}}$,
$B_i(k) \cap \Gamma_{\mathbb{Q}}$ dense in B_i.

That's the "general case of S. Lang's conjecture."
Proved by Faltings (ab. varieties, <u>f.g. groups (Γ)</u>, <u>curves</u> 1983)
"Mordell conjecture"

Another proof by Vojta 1991
A proof for every V (still just for Γ) by Faltings 1991
General case Faltings 1999, $k = \overline{\mathbb{Q}}$.
<u>tori</u>: Laurent, Liardet, Hindry
<u>semiab</u>: Vojta, McQuillan

<u>Char $p > 0$?</u>
<u>Th.</u> $k_0 \subset k$ extension of alg. closed fields (of char. $p > 0$),
A/k semiab. variety, $\Gamma \subset A(k)$ f.g. group, $V \subset A$ irreducible closed subvariety s.t. $V(k) \cap \Gamma_{p'}$ is dense.

(continued, next page)

Then, there exists:

- a semiabelian variety A_0/k_0
- a semiabelian variety $A_1 \subset A$
- $V_0 \subset A_0$
- a surjective morphism $h: A_1 \to (A_0)_k$
- a point $a \in A(k) \cap \Gamma_{p'}$

such that $V = a + h^{-1}((V_0)_k)$

Conjectured by Abramovich–Voloch
Proved by Hrushovski, Buium (char $p = 0$) "opening move" to Hrushovski's proof
curves Manin, Grauert, Samuel (char p)
Moosa–Scanlon over $\overline{\mathbb{F}_p}$
Rössler–Corpet, $p > 0$,
The Γ case follows from Manin-Mumford, $\{0\}_{\mathbb{Q}}$. ← Pink, Rössler

5. Stating the André-Oort conjecture

a) Hermitian symmetric domains

M complex manifold ($M \overset{\text{open}}{\subset} \mathbb{C}^n$)

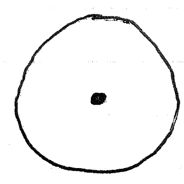
hermitian manifold: hermitian scalar products on the complex tangent spaces, which vary smoothly with the point

symmetric: for every $p \in M$ there is an isometric involution S_p of M of which p is an isolated fixed point.

Example

$M \subset \mathbb{C}^n$ open

$(h_{ij}(z))$



$\{|z| < 1\}$ with the hyperbolic metric $\|dz\|^2 = \dfrac{|dz|^2}{(1-|z|^2)^2}$

$z \mapsto -z$ at $p = 0$.

$\Rightarrow G = \begin{pmatrix}\text{group of holomorphic isometries} \\ \text{of } M\end{pmatrix}$

is a real Lie group.

$G^+ =$ conn. comp. of Id_M in G.

For every $p \in M$, $G^+ \to M$ is surjective

$g \mapsto gp$

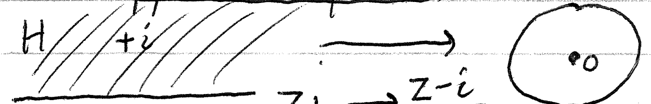
$K_p = \mathrm{Fix}(p)$ is a compact subgroup of G^+, $M \simeq G^+/K_p$

Generalization: <u>bounded symmetric domains</u>

$M \subset \mathbb{C}^n$ open + symmetries

$\rightsquigarrow$ the hermitian structure comes out automatically.

<u>Poincaré upper half-plane</u>



$z \longmapsto \dfrac{z-i}{z+i}$

<u>Siegel upper half space</u>

$\left\{ X+iY \;\middle|\; \begin{matrix} \text{symmetric} \\ Y>0 \end{matrix} \right\} \longrightarrow$ bounded domain $\{ I_g - Z^* Z > 0,\ Z \text{ symmetric} \}$

$Z \longmapsto (Z - iI_g)(Z + iI_g)$

<u>Classification</u> according to curvature (of Hermitian symmetric domains)

- $\boxed{\lneq 0 \text{ "of non compact type"}}$
- $=0$ $\quad \mathbb{C}^n/\text{discrete subgroups}$
- >0 $\quad \mathbb{P}^1(\mathbb{C})$

M hermitian symmetric domain of non compact type

$\Gamma \subset \mathrm{Isom}^+(M)$ arithmetic subgroup

$\exists$ alg. group $G/\mathbb{Q} \hookrightarrow GL(n)$

a surjective morphism $G(\mathbb{R})^+ \xrightarrow{\rho} \mathrm{Isom}^+(M)$ with compact kernel

s.t. $\rho^{-1}(\Gamma) \underset{\text{commensurable}}{\sim} GL(n,\mathbb{Z}) \cap G(\mathbb{Q})$

$\Rightarrow \Gamma$ is discrete

Γ has a torsion free subgroup of finite index

<u>Thms</u> (A) <u>Baily–Borel</u> ('66) $\exists$ a <u>quasi-projective alg. variety</u> $V_\Gamma/\mathbb{C}$ and a biholomorphic $M/\Gamma \to V_\Gamma(\mathbb{C})$ (Γ torsion free)

(B) <u>Borel</u> (1972) For every smooth alg. variety $V/\mathbb{C}$ and every holomorphic map $V(\mathbb{C}) \to V_\Gamma(\mathbb{C})$, this map is algebraic.

$\Rightarrow M/\Gamma$ is canonically an alg. variety.

<u>Shimura</u> (G, M) $\quad \boxed{G^{ad} = G/Z(G)}$

algebraic group — hermitian symmetric domain of non-compact type

$G^{ad}(\mathbb{R})^+ \xrightarrow{\sim} \mathrm{Isom}^+(M)$

$\Gamma \subset G(\mathbb{Q})$ arithmetic subgroup $(\sim$ "$G(\mathbb{Z})$"$)$
V_Γ has a natural $\overline{\mathbb{Q}}$-structure

<u>Special points</u>: $p \in M$. For every $z \in S_1 = \{|z| = 1\}$. $\exists\, u_p(z) \in \mathrm{Isom}^+(M)$ which fixes p and acts by rotation by z on T_pM

$$T_p(u_p(z)) = z\,\mathrm{Id}$$

$MT(p)$ = smallest alg. subgroup of G defined over $\mathbb{Q}$ whose image in $\mathrm{Isom}^+(M)$ contains $u_p(z)$. For all z.

p is a special point iff $MT(p)$ is commutative.

<u>Shimura subvariety</u> Given by $(G', M') \longrightarrow (G, M)$

$G' \xrightarrow{f} G$, $M' \xrightarrow{\varphi} M$ compatible

$M'/\Gamma' \longrightarrow M/\Gamma$ if the image of Γ' is contained in Γ.

Add action of the Hecke correspondences

$$\begin{tikzcd}
M \arrow[rr, "\cdot g"] \arrow[dd] \arrow[dr, dash] & & M \arrow[dd] \arrow[dl, dash] \\
& M/\Gamma \cap g\Gamma g^{-1} \arrow[dl] \arrow[dr] & \\
M/\Gamma & & M/\Gamma
\end{tikzcd} \qquad g \in G(\mathbb{Q})$$

Irred. special subvarities = irreducible components of translates by Hecke corresp. of Shimura subvarieties.

Now state André-Oort by general principles stated earlier!