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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Alex Kruckman **Email/Phone:** Kruckman@gmail.com

Speaker's Name: Pierre Simon

Talk Title: An Introduction to Stability-Theoretic Techniques (III)

Date: 02/05/14 **Time:** 11:00 am / pm (circle one)

List 6-12 key words for the talk: NIP theories, o-minimality, cell decomposition, VC dimension, generically stable types

Please summarize the lecture in 5 or fewer sentences: Part 3 of 3. This talk focused on NIP theories, a class of theories extending the stable theories. VC-dimension, examples, and a characterization via indiscernible sequences. O-minimal theories are a class of ordered (hence unstable) NIP theories with tame structure: cell decomposition, uniform finiteness. At the end of the talk, a return to definable types - product types and generically stable types by example in ACVF.

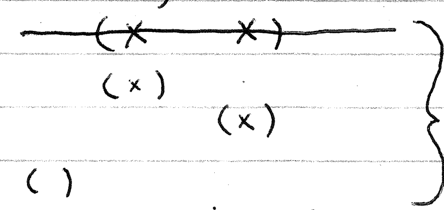
CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

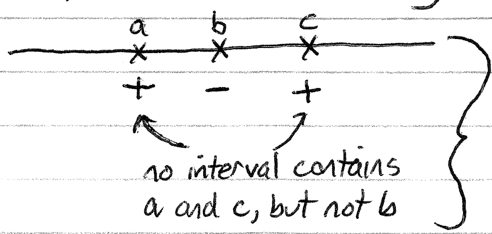
- ☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - <u>**Computer Presentations**</u>: Obtain a copy of their presentation
  - <u>**Overhead**</u>: Obtain a copy or use the originals and scan them
  - <u>**Blackboard**</u>: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - <u>**Handouts**</u>: Obtain copies of and scan all handouts
- ☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack <u>with this completed sheet on top</u> and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☒ Email the re-named files to <u>notes@msri.org</u> with the workshop name and your name in the subject line.

<u>Notation:</u> $A \subseteq X$, $\mathcal{C} \subseteq \mathcal{P}(X)$,
$\mathcal{C} \cap A = \{ C \cap A \mid C \in \mathcal{C} \}$

$X = \mathbb{R}$, $\mathcal{C}$ = intervals — VC-dimension 2

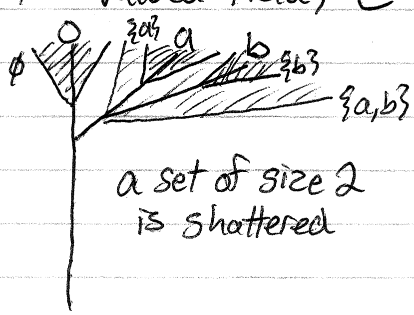


a set of size 2 is shattered

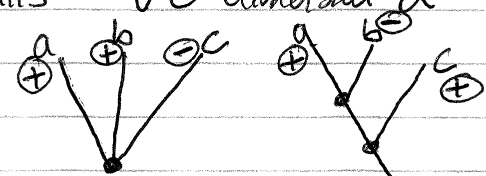


a set of size 3 is not shattered

X = valued field, $\mathcal{C}$ = balls — VC-dimension 2



a set of size 2 is shattered



a set of size 3 is not

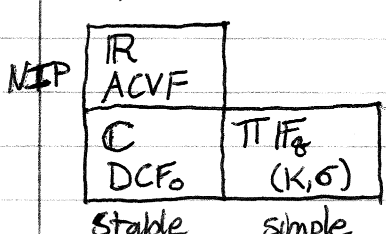
<u>$x \mid y$ in $Th(\mathbb{N})$ has IP:</u>

$p_1, \dots, p_n$ primes

For $J \subseteq \{1, \dots, n\}$, let $b_J = \prod_{i \in J} p_i$

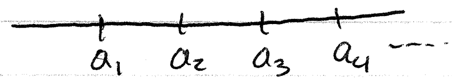
Then $p_i \mid b_J$ iff $i \in J$.

<u>A picture of the universe:</u>



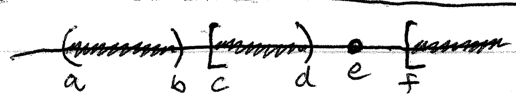
Indiscernible sequences

Ex: $(\mathbb{Q}, \leq)$ $A = \emptyset$



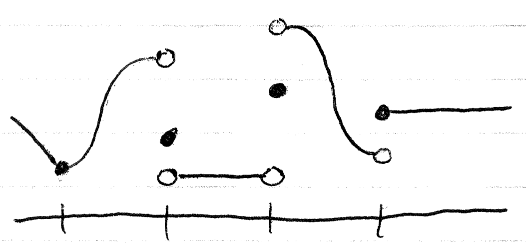
What does it mean to say the sequence of types $(tp(a_i/M) : i \in \omega)$ converges?
For all $\varphi(x) \in \mathcal{L}_M$ (basic open sets in the type space),
$\exists N_\varphi$ s.t. $(\forall i > N_\varphi,\ \varphi(x) \in tp(a_i/M))$ or $(\neg\varphi(x) \in tp(a_i/M)\ \forall i > N_\varphi)$
No infinite alternation of the truth of φ on $(a_i : i \in \omega)$.

Definable subsets of M o-minimal



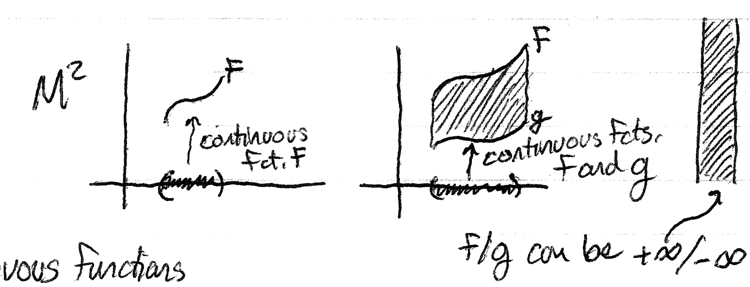
$a, b, c, d, e, f \in M$

Definable functions in o-minimal theories



Cells

dim 0 • points
dim 1 ⟵⟶ intervals



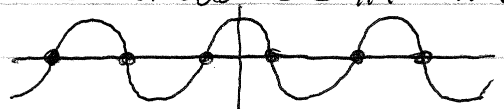
Cells in M^{n+1} are graphs of continuous functions or regions between graphs of continuous functions on cells in M^n.

In $(\mathbb{R}, +, \cdot, -, 0, 1)$, the set defined by $p(x) \leq 0$, p a polynomial, is a finite union of points and intervals. $QE \Rightarrow \mathbb{R}$ is o-minimal.

o-minimal expansions of $\mathbb{R}$

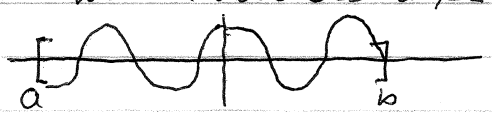
$\mathbb{R}_{exp}$: $(\mathbb{R}, +, -, \cdot, 0, 1)$ together with $\exp: \mathbb{R} \to \mathbb{R}$

We cannot add $\cos: \mathbb{R} \to \mathbb{R}$



$\{x \mid \cos(x) = 0\}$ is an infinite discrete set

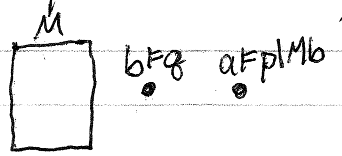
But we can add $\cos: [a,b] \to \mathbb{R}$ for any bounded interval $[a,b]$.



More generally, $B: \prod_{i=1}^{n} [a_i, b_i] \subset \mathbb{R}^n$, $f: U \to \mathbb{R}$ analytic on $U \supseteq B$ open. Adding $f|_B$ preserves o-minimality.

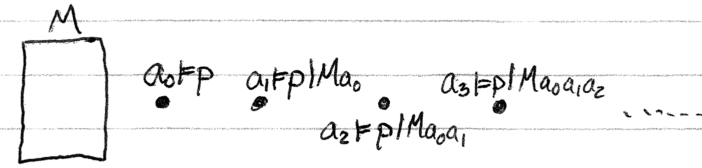
Product types $(\in S(M))$

Recall if p is a type over M definable in M and $M \subseteq B$, we define $p \mid B \in S(B)$ by using the defining scheme for p.

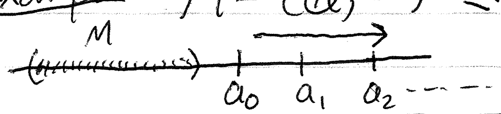


$p \otimes q = tp(a, b/M)$

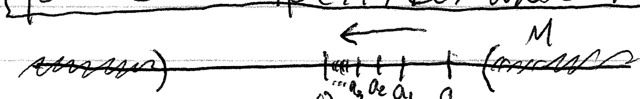
Morley sequence:



Examples $M = (\mathbb{Q}, \leq) \preceq \mathcal{U}$, $p = \text{"}+\infty\text{"} \in S(M)$

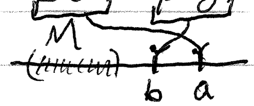


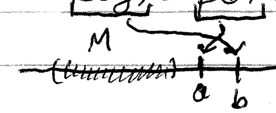
$p = \text{"}0^+\text{"} = tp(1/t / \mathbb{Q})$ where $t > \mathbb{Q}$



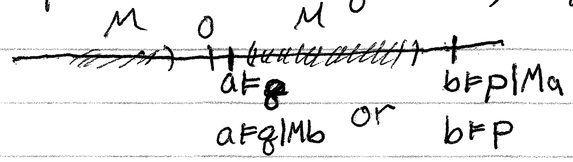
Taking $p = +\infty$ again, $p(x) \otimes p(y) \vdash x > y$ $\qquad$ $p(y) \otimes p(x) \vdash x < y$

p does not commute with itself





If $p = +\infty$ and $q = 0^+$, $p(x) \otimes q(y) = q(y) \otimes p(x)$

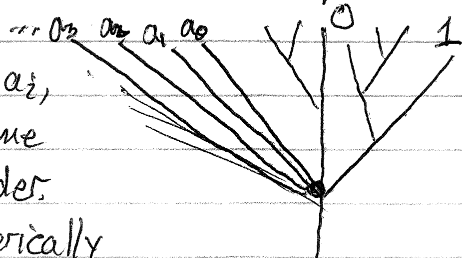


$K \models ACVF$

$\mathcal{O} \subseteq B^{cl}_0(0)$ Look at $p_{\mathcal{O}}(x)$. $val(x) = 0 \in p_{\mathcal{O}}(x)$

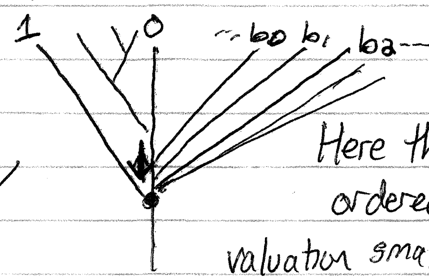
$val(x-a) = 0 \in p_{\mathcal{O}}(x)$

$\forall a \in \mathcal{O}(K)$



permuting the a_i, we get the same thing: no order.

$p_{\mathcal{O}}(x)$ is generically stable = stably dominated in ACVF



Now look at $p_{\mathfrak{m}}(x)$.

$0 < val(x) < \gamma$ for all $\gamma \in \Gamma_{>0}(K)$

This type is *not* generically stable/stably dominated.

Here the b_i are ordered. Each has valuation smaller than the last.

Introduction to model theoretic techniques

Pierre Simon

Université Lyon 1, CNRS and MSRI

Introductory Workshop: Model Theory, Arithmetic Geometry and Number Theory

VC-dimension

Let X be a set and $\mathcal{C} \subseteq \mathfrak{P}(X)$ a family of subsets of X.
Let $A \subseteq X$, then $\mathcal{C}$ *shatters* A if $\mathcal{C} \cap A = \mathfrak{P}(A)$.

Definition

The family $\mathcal{C}$ has *VC-dimension* d if it shatters some subset $A \subseteq X$ of size d, but no subset of size $d+1$.
If $\mathcal{C}$ shatters subsets of arbitrary large (finite) size, we say that it has infinite VC-dimension.

Examples: The family of intervals of $(\mathbb{R}, \leq)$ has VC-dimension 2.
The family of half-spaces of $\mathbb{R}^2$ has VC-dimension 3.

Define the *shatter function* $\pi_{\mathcal{C}}$ of $\mathcal{C}$ as

$$\pi_{\mathcal{C}}(n) = \max_{A \subseteq X, |A| \leq n} |\mathcal{C} \cap A|.$$

Note that $\pi_{\mathcal{C}}(n) = 2^n$ if and only if VC-dim$(\mathcal{C}) \geq n$.

Fact (Sauer-Shelah lemma)

Either :

• $\pi_{\mathcal{C}}(n) = 2^n$ *for all* n *(infinite VC-dimension)*

or

• $\pi_{\mathcal{C}}(n) = O(n^d)$ *(one can take* $d =$*VC-dim*$(\mathcal{C})$*).*

The *VC-density* of $\mathcal{C}$ defined as the infimum of r such that $\pi_{\mathcal{C}}(n) = O(n^r)$ is often more meaningful than the VC-dimension.

NIP theories

Let M be a structure and $T = Th(M)$.

$$\varphi(\bar{x};\bar{y}) \longrightarrow \mathcal{C}_\varphi = \{\varphi(M;\bar{b}) : \bar{b} \in M^{|\bar{y}|}\} \subseteq \mathfrak{P}(M^{|\bar{x}|}).$$

Definition

The formula $\varphi(\bar{x};\bar{y})$ is *NIP* (No Independence Property) if the family $\mathcal{C}_\varphi$ has finite VC-dimension.
The theory T is NIP if all formulas are.

In other words, the formula $\varphi(\bar{x};\bar{y})$ has IP if for all n, one can find $\bar{a}_1, \ldots, \bar{a}_n \in M^{|\bar{x}|}$ and a family $(\bar{b}_J : J \in \mathfrak{P}(\{1, \ldots, n\}))$ such that:

$$M \models \varphi(\bar{a}_i; \bar{b}_J) \iff i \in J.$$

Examples:

- The formula $x \leq y$, where $\leq$ is a linear order is NIP;
- The formula $x|y$ (x divides y) in $\mathbb{N}$ has IP.
- Every stable theory is NIP;
- $Th(\mathbb{R}; 0, 1, +, -, *, \leq)$ is NIP;
- Some theories of valued fields: ACVF, $Th(\mathbb{Q}_p)$ are NIP.

Lemma (VC-duality)

A formula $\varphi(\bar{x}; \bar{y})$ is NIP if and only if the opposite formula $\varphi^{opp}(\bar{y}; \bar{x})$ is NIP.

Indiscernible sequences

Definition

Let $(I, <_I)$ be a linear order and $A \subset M$. A sequence $(a_i : i \in I)$ of tuples of M is *indiscernible* over A if for all $i_1 <_I \cdots <_I i_k$ and $j_1 <_I \cdots <_I j_k$, we have

$$\mathrm{tp}(a_{i_1} \ldots a_{i_k}/A) = \mathrm{tp}(a_{j_1} \ldots a_{j_k}/A).$$

Fact (Ramsey+Compactness)

Given any sequence $(a_i : i < \omega)$ *of tuples and a linear order* $(I, <_I)$*, there is an indiscernible sequence* $(b_i : i \in I)$ *in* $\mathcal{U}$ *such that for any* $i_1 <_I \cdots <_I i_k$ *if*

$$\mathcal{U} \models \varphi(b_{i_1}, \ldots, b_{i_k}),$$

then there are $j_1 < \cdots < j_k < \omega$ *such that*

$$\mathcal{U} \models \varphi(a_{j_1}, \ldots, a_{j_k}).$$

Lemma

T is NIP if and only if for any indiscernible sequence $(a_i : i < \omega)$ and any model M, the sequence of types $(\mathrm{tp}(a_i/M) : i < \omega)$ converges.

More generally:

Lemma

The theory T is NIP if and only if for any set $A \subseteq \mathcal{U}$, any sequence of types over A has a converging subsequence.

Theorem

If all formulas $\varphi(x; \bar{y})$, x a singleton, are NIP, then T is NIP.

o-minimality

Assume that the language L contains a distinguished binary relation $\leq$ which defines a linear order on M.

Definition

The structure $(M, \leq, \ldots)$ is o-minimal if any definable subset of M is a finite union of intervals and points.

Fact

Assume that M is o-minimal, $a, b \in M \cup \{\pm\infty\}$ and let $f : (a, b) \to M$ be a definable function, then there are

$$a = a_0 < a_1 < \cdots < a_k = b$$

such that for each i, $f|_{(a_i, a_{i+1})}$ is either constant or a continuous monotonic bijection to an interval.

Fact (Cell decomposition)

Assume that M is o-minimal, then any definable subset of M^k is a finite union of cells.

Uniform finiteness

Fact

Let M be o-minimal. Let $\phi(x, \bar{y})$ be a formula, then there is some integer n such that any $\phi(x, \bar{b})$, $\bar{b} \in M$, defines a union of at most n intervals.

Corollary

Assume that M is o-minimal, then any structure elementarily equivalent to M is o-minimal. Hence o-minimiality is a property of the theory $Th(M)$.

Examples of o-minimal structures

- $\mathbb{R}$, with the field structure;
- $\mathbb{R}_{exp}$: the field $\mathbb{R}$ with the exponential function;
- $\mathbb{R}_{an}$: the field $\mathbb{R}$ along with restricted analytic functions;
- $\mathbb{R}_{an,exp}$.

Back to definable types

Let $p \in S_{\bar{x}}(\mathcal{U})$ be definable over a model $M \prec \mathcal{U}$. Recall that this means that we have a mapping

$$\varphi(\bar{x};\bar{y}) \longrightarrow d_p\varphi(\bar{y}), \qquad d_p\varphi(\bar{y}) \in L_M$$

such that for all $\bar{b} \in \mathcal{U}^{|\bar{y}|}$;

$$\varphi(\bar{x};\bar{b}) \in p \iff \mathcal{U} \models d_p\varphi(\bar{b}).$$

Product of definable types

Let $p(x)$ and $q(y)$ in $S(M)$ be definable, then one can define the product $p \otimes q(x, y)$ as $\mathrm{tp}(a, b/M)$, where

$$b \models q \text{ and } a \models p|Mb.$$

A Morley sequence of p over M is a sequence $(a_i : i < \omega)$ such that:

$$a_0 \models p \restriction M \qquad a_{k+1} \models p \restriction Ma_0 ... a_k.$$

Such a sequence is indiscernible over M.

Generically stable types

Definition

A type $p \in S(M)$ is *generically stable* if:
• p is definable;
• some/any Morley sequence $(a_i : i < \omega)$ of p is totally indiscernible (*i.e.*, every permutation of it is indiscernible).

Fact

A generically stable type commutes with any definable type.

Example: (ACVF) the generic type of a closed ball.

End of talk 3.