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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Alex Kruckman **Email/Phone:** kruckman@gmail.com

Speaker's Name: Margaret Thomas

Talk Title: The Pila-Wilkie Theorem and variations

Date: 02/05/14 **Time:** 12:00 am / pm (circle one)

List 6-12 key words for the talk: Pila-Wilkie, counting rational points, o-minimality, cell decomposition, heights

Please summarize the lecture in 5 or fewer sentences: Slides (1-16) with some supporting boardwork. The Pila-Wilkie theorem on counting points (by rational height) on definable sets in o-minimal structures, and its proof (with some improvements from recent notes by Alex Wilkie).

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- [x] Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- [x] Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
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- [x] For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- [x] When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- [x] Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

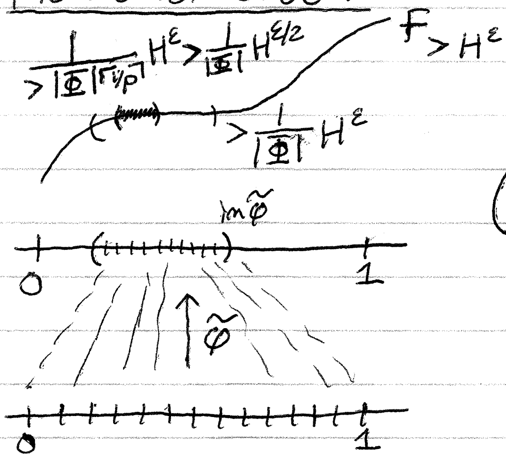
Height of rational tuples

If $\gcd(a,b)=1$, $ht\left(\frac{a}{b}\right) = \max\{|a|, |b|\}$

$$ht(q_1, \ldots, q_n) = \max_{1\le i \le n} \{ ht(q_i) \}$$

Notation: $\mathbb{Q}(H) = \mathbb{R}(\mathbb{Q}, H) = \{ q \in \mathbb{Q} \mid ht(q) \le H \}$

Picture for Slide 7:



$$\begin{array}{cc} (\tilde{\varphi}, & f \circ \tilde{\varphi}) \\ \parallel & \parallel \\ \varphi & \psi \end{array}$$

The Pila–Wilkie Theorem and Variations

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Model Theory, Arithmetic Geometry and Number Theory
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Pila–Wilkie Theorem

Let $S \subseteq \mathbb{R}^n$ be definable in an o-minimal expansion of the ordered field of real numbers. Assume that S contains no infinite semialgebraic subset*. Let $\varepsilon > 0$. There exists $C = C(\varepsilon) > 0$ such that if $H \geq C$, then S contains at most H^ε rational points of height at most H, i.e. setting $S(\mathbb{Q}, H) := \{\bar{q} \in S \cap \mathbb{Q}^n \mid \operatorname{ht}(\bar{q}) \leq H\}$, we have that for $H \geq C$, $|S(\mathbb{Q}, H)| \leq H^\varepsilon$.

*If we set
$S^{\text{alg}} :=$ union of infinite, connected semialgebraic subsets of S, and $S^{\text{trans}} := S \setminus S^{\text{alg}}$, then Pila and Wilkie in fact proved a stronger statement, in which this assumption on S is dropped, and the conclusion states that for $H \geq C$, we have $|S^{\text{trans}}(\mathbb{Q}, H)| \leq H^\varepsilon$.

- We may assume $S \subseteq (0,1)^k$ (consider maps $x \mapsto \pm x^{\pm 1}$ which preserve definability and height).
- We may assume that $S = \Gamma(F)$ for some definable function $F\colon (0,1)^m \to (0,1)$ (using cell decomposition and $S = S^{\text{trans}}$).
- The key auxiliary result is the following:

Reparameterization Lemma

Let $F\colon (0,1)^m \to (0,1)^n$ be definable. For all $p \geq 1$, there exists a finite set Φ of C^p maps $\phi\colon (0,1)^m \to (0,1)^m$ such that

- $\bigcup_{\phi \in \Phi} \mathrm{Im}(\phi) = (0,1)^m$;
- for all $\phi \in \Phi$, $\left|\left|\phi^{(\alpha)}\right|\right|, \left|\left|(F \circ \phi)^{(\alpha)}\right|\right| \leq 1$, for all $|\alpha| \leq p$.

Moreover, $|\Phi|$ depends on p and uniformly on F, as do the $\phi \in \Phi$.

Assuming Reparameterization, sketch proof of Pila–Wilkie for curves, i.e.

Theorem

Let $f\colon (0,1) \to (0,1)$ be definable and assume that $\Gamma(f) = \Gamma(f)^{trans}$. For all $\varepsilon > 0$, there exists $C = C(\varepsilon) > 0$ such that, for all $H > C$, $|\Gamma(f)(\mathbb{Q},H)| \le H^{\varepsilon}$.

Observe that for all subintervals $I \subseteq (0,1)$ and all non-zero $P \in \mathbb{R}[X,Y]$, there is some $\alpha \in I$ such that $P(\alpha, f(\alpha)) \neq 0$.

There are 3 steps to the proof.

STEP 2 is purely number theoretic.
We start by choosing $p, d \in \mathbb{N}$ satisfying certain easy conditions ($p \geq 25$, $4p \leq d^2 \leq 5p$).
For C^p functions $\phi, \psi\colon (0,1) \to (0,1)$, whose derivatives up to order p are bounded by 1, set, for $c_{s,t} \in \mathbb{Z}$,

$$G(x) := \sum_{0 \leq s,t \leq d-1} c_{s,t} \phi(x)^s \psi(x)^t.$$

Suppose H is as large as you need (wrt d) and suppose $\beta \in (0,1)$ is such that $\phi(\beta), \psi(\beta) \in \mathbb{Q}(H)$. Then $G(\beta) = 0$ or $|G(\beta)| \geq \frac{1}{H^{2(d-1)}}$.
Goal: choose $c_{s,t} = c_{s,t}(H,d)$ and $l = l(H,d)$ such that if $I_{H,d} \subseteq (0,1)$ is an interval of length at most l, and $\beta \in I_{H,d}$ has $\phi(\beta), \psi(\beta) \in \mathbb{Q}(H)$, then $|G(\beta)| < \frac{1}{H^{2(d-1)}}$.

Apply Taylor's Theorem around some $\alpha \in (0,1)$:

$$G(x) := \sum_{0 \leq s,t \leq d-1} c_{s,t} \left(\sum_{j=0}^{p-1} \frac{(\phi^s \psi^t)^{(j)}(\alpha)}{j!}(x-\alpha)^j + \frac{(\phi^s \psi^t)^{(p)}(\xi_{s,t})}{p!}(x-\alpha)^p \right)$$

for $\xi_{s,t}$ between x and α.

Now use the fact that $\left|\left|\phi^{(i)}\right|\right|, \left|\left|\psi^{(i)}\right|\right| \leq 1$ for all $i = 0, \ldots, p$, and Thue-Siegel/ Dirichlet Box Principle, to find $c_{s,t}$ and l such that if $\beta \in (\alpha - \frac{l}{2}, \alpha + \frac{l}{2})$, then $|G(\beta)| < \frac{1}{H^{2(d-1)}}$. Hence for those β for which also $\phi(\beta), \psi(\beta) \in \mathbb{Q}(H)$, we have $G(\beta) = 0$.

l and d are related in such a way that increasing the length of the interval increases the degree of d required.

STEP 1 Start with $f\colon (0,1) \to (0,1)$ as above, $\varepsilon > 0$, and suppose for a contradiction that, for infinitely many H, $|\Gamma(f)(\mathbb{Q},H)| > H^{\varepsilon}$. Choose $d = d(\varepsilon), p = p(\varepsilon)$ as proscribed above so that $l(d,H) > \frac{2}{H^{\frac{\varepsilon}{2}}}$.

By Reparameterization, there exists $\Phi(p(\varepsilon))$, a finite set of C^p functions, such that $\bigcup_{\phi \in \Phi} \mathrm{Im}(\phi) = (0,1)$ and for all $\phi \in \Phi$, $\left|\left|\phi^{(\alpha)}\right|\right|, \left|\left|(F \circ \phi)^{(\alpha)}\right|\right| \leq 1$, for all $|\alpha| \leq p$.
Since $\{\mathrm{Im}(\phi)\}_{\phi \in \Phi}$ cover $(0,1)$, by the Pigeonhole Principle (PHP), there is some $\tilde{\phi} \in \Phi$ for which $\Gamma(f\restriction_{\mathrm{Im}(\tilde{\phi})})(\mathbb{Q},H) > \frac{1}{|\Phi|} H^{\varepsilon}$.

Now cover $\mathrm{dom}(\tilde{\phi}) = (0,1)$ with $\lceil \frac{1}{l} \rceil$ intervals of length at most l. Again, by PHP, one of these subintervals I is such that $\Gamma(f\restriction_{\mathrm{Im}(\tilde{\phi}\restriction_I)})(\mathbb{Q},H) > \frac{1}{|\Phi|\lceil \frac{1}{l} \rceil} H^{\varepsilon} > \frac{1}{|\Phi|} \cdot \frac{l}{2} \cdot H^{\varepsilon} > \frac{1}{|\Phi|} H^{\frac{\varepsilon}{2}}$.

STEP 3 Taking $\phi = \tilde{\phi}$ and $\psi = f \circ \tilde{\phi}$ in Step 2, we see that, for infinitely many H, there is a function

$$\widetilde{G_H}(x) := \sum_{0 \le s,t \le d-1} c_{s,t}(\varepsilon, H) x^s f(x)^t$$

such that $\left| Z(\widetilde{G_H}) \right| > \frac{1}{|\Phi|} H^{\frac{\varepsilon}{2}}$.

But consider the definable family

$$\mathscr{F} := \{ \ \{x \mid \sum_{0 \le s,t \le d-1} r_{s,t} x^s f(x)^t = 0\} \mid r_{s,t} \in \mathbb{R} \text{ for } 0 \le s,t \le d-1\}.$$

Each member of $\mathscr{F}$ has only finitely many connected components, and we can bound this number uniformly, by $N(d)$, say; since $\Gamma(f) = \Gamma(f)^{\mathrm{trans}}$, these connected components must be singletons.

So $\left| Z(\widetilde{G_H}) \right| \le N(d(\varepsilon))$.

Now let $H > (N(d(\varepsilon)) |\Phi|)^{\frac{2}{\varepsilon}}$. Contradiction. $\square$

General case: consider $\Gamma(F)$, for $F\colon (0,1)^m \to (0,1)$ with $\Gamma(F) = \Gamma(F)^{\text{trans}}$.
Step 1 and Step 2 go through routinely to give us an analogous polynomial $P_H \in \mathbb{R}[X_1,\ldots,X_{m+1}]$ with $Z(P_H) \cap \Gamma(F)(\mathbb{Q},H) \geq H^{\varepsilon^r}$, for some $r(m) >> 0$.
Note that $\Gamma(F) = \Gamma(F)^{\text{trans}} \Rightarrow \dim(Z(P_H) \cap \Gamma(F)) < m$ (else contains $\Delta_H \cap \Gamma(F)$ for some open box Δ_H of dim $m+1$).
We would like to employ an argument which uses induction on dimension to get a contradiction at this point, but the definition of the above set depends on H.
However this just means we were proving the wrong theorem.

Uniform Pila–Wilkie Theorem

Let $\{S_{\bar{x}} \mid \bar{x} \in \mathbb{R}^k\}$ be a definable family, $S_{\bar{x}} \in \mathbb{R}^n$ for all $\bar{x} \in \mathbb{R}^k$. For all $\varepsilon > 0$, there exists $D = D(\varepsilon) > 0$ such that, for all $\bar{x} \in \mathbb{R}^k$ and $H \geq D$,

either $(1)_{\bar{x}}$ $S_{\bar{x}}$ contains an infinite semialgebraic subset

or $(2)_{\bar{x}}$ $|S_{\bar{x}}(\mathbb{Q}, H)| < H^{\varepsilon}$.

Moreover, which of $(1)_{\bar{x}}$ or $(2)_{\bar{x}}$ holds depends definably on $\bar{x}$, and, if $(1)_{\bar{x}}$ holds, the set may be chosen to depend definably on $\bar{x}$.

The same strategy works for proving this, as all arguments are uniform in definable families and the sets $(Z(P_H) \cap \Gamma(F))$ lie in one fixed family not depending on H.

C^1-1

C^1-reparameterization for $F=(F_1,\ldots,F_n)\colon (0,1)\to(0,1)^n$.
Assume without loss that one F_i is the identity.
Subdivide $(0,1)$ into intervals on which each F_j is C^1 and $|F'_j|-|F'_k|$ has constant sign, for $j,k=1,\ldots,n$.
(Monotonicity Theorem and Uniform Bounds)
On subinterval I choose the j_I such that $\left|F'_{j_I}\right|$ is biggest on I (it will be ≥ 1). Set $\phi_I(x)=F_{j_I}^{-1}(c+(d-c)x)$, where $F_{j_I}=(c,d)$.
The required parameterization is the set of ϕ_Is together with constant maps for the singletons separating Is.

$C^p - 1$

C^p-repara. for $F\colon (0,1) \to (0,1)^n$.

It is enough to find a "C^p-reparameterization" whose derivatives are bounded by some function of p.

Subdivide $(0,1)$ into intervals I on which all F_i are C^p, $|F_i'| \leq 1$, and the coordinate functions of $F^{(k)}$ are either identically zero or nowhere zero. (Monotonicity Theorem and $C^1 - 1$-reparam.)

For each $I = (a,b)$, set $\phi_I = a + \frac{1}{2}(b-a)x^p$.

Then $\left|\left|\phi_I^{(q)}\right|\right| \leq p!$, for $0 \leq q \leq p$.

Now consider the derivatives of $F \circ \phi_I$. These are expressions in terms of $\phi_I^{(q)}$s and $f^{(q)} \circ \phi_I$, so we need to bound the latter.

(New) Analytic Trick

Lemma (no model theory; only analysis)

Let $p \geq 1$, I a bounded interval in $\mathbb{R}$, $f: I \to (0,1)$ a C^{p+1} function such that for all $x \in I$ and $j = 0, \ldots, p+1$, $f^{(j)}(x) \neq 0$. Then for all $x \in I$ and $j = 0, \ldots, p$, $\left|f^{(j)}(x)\right| < \left(\frac{j+1}{\delta_I(x)}\right)^j$, where $\delta_I(x)$ is the distance from x to the nearest endpoint of I.

As $\phi_I = a + \frac{1}{2}(b-a)x^p$ maps onto $(a, \frac{b+a}{2})$, $\delta_I(\phi_I(x)) = \frac{1}{2}(b-a)x^p$, and so, applying the lemma to $F\restriction_I$, it comes, after some computations, that $\left|\left|(F \circ \phi_I)^{(q)}\right|\right| \leq c_1 p^{c_2 p}$, for $0 \leq q \leq p$.

C^p-repara. for $F\colon (0,1)^m \to (0,1)^n$.
We may assume F is C^p on $(0,1)^m$. (Cell Decomposition Theorem)
First, induction on m with x_m as a parameter to obtain what is almost a C^p-reparameterization: a finite set Φ_0 of functions ϕ which cover $(0,1)^m$ such that $\left|\left|\phi^{(\alpha)}\right|\right|, \left|\left|(F \circ \phi)^{(\alpha)}\right|\right| \leq 1$, *for all* $\alpha \in \mathbb{N}^m$ *with* $|\alpha| \leq p$ *AND* $\alpha_m \leq 0$.
Now an induction on k (where the above is the $k = 0$ case) to obtain analogously defined Φ_{k+1} (where $\alpha_m \leq k+1$) from Φ_k ($\alpha_m \leq k$).
This uses a similar (but much less messy) substitution lemma to the one from the original proof, where one builds a function by taking as coordinate functions all ϕ for $\phi \in \Phi_{k+1}$ as well as the derivatives of all $F \circ \phi$, and their derivatives, and then one reparameterizes and substitutes in the (domain of the) last variable.

The natural extension of these ideas is to consider algebraic points instead of rational points (using the absolute multiplicative height).

That could mean either points with coordinates in a fixed real number field $F \subseteq \mathbb{R}$ of degree k.
We count points in the analogously defined $S^{\text{trans}}(F,H)$.

Or it could mean algebraic points whose coordinates have degree bounded by a fixed number k.
In that case we count the size of $|S^{\text{trans}}(k,H)| = |S^{\text{trans}} \cap \{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid \text{for all } i, [\mathbb{Q}(x_i):\mathbb{Q}] \leq k \text{ and } \mathrm{ht}(x_i) \leq H\}|$.

In both cases, the analogous version of the Pila-Wilkie Theorem holds (Pila 2009).

It is not possible to obtain an improvement in the H^{ε} bound which would hold for all o-minimal expansions of the real ordered field.

Given any function $\varepsilon(H) \to 0$ as $H \to \infty$, there is a transcendental analytic function $f\colon [0,1] \to \mathbb{R}$ and a sequence $(H_n)_n$ with $H_n \to \infty$ such that, for all $n \in \mathbb{N}, |\Gamma(f)(\mathbb{Q}, H_n)| \geq H_n^{\varepsilon(H_n)}$. These functions are definable in the o-minimal structure $\mathbb{R}_{\mathrm{an}}$.

However, there is a proposed improvement for $\mathbb{R}_{\exp}$:

Wilkie's Conjecture (2006)

Let $F \subseteq \mathbb{R}$ be a number field of degree k. Suppose S is definable in $\mathbb{R}_{\exp}$ and does not contain an infinite semialgebraic subset. There exist $c(S,k), \gamma(S) > 0$ such that $|S(F,H)| \leq c(\log H)^{\gamma}$.

There is a version for algebraic points of bounded degree formulated by Pila (2010), where the exponent $\gamma = \gamma(S,k)$.

Wilkie's Conjecture (2006, in the form stated by Pila in 2010)

1. Let $F \subseteq \mathbb{R}$ be a number field of degree $k \in \mathbb{N}$. For all sets S definable in $\mathbb{R}_{\exp}$, there exist $c(S,k), \gamma(S) > 0$ s.t. $|S^{\text{trans}}(F,H)| \leq c(\log H)^{\gamma}$, for $H \geq e$.
2. Let $k \in \mathbb{N}$. For all sets S definable in $\mathbb{R}_{\exp}$, there exist $c(S,k), \gamma(S,k) > 0$ s.t. $|S^{\text{trans}}(k,H)| \leq c(\log H)^{\gamma}$, for $H \geq e$.

What do we already know?

- (1) holds for all S with $\dim(S) = 1$ (Jones-T./Butler (2010)).
- Goes via proving the bound of (1) for $S = \Gamma(f)$, where f is a one variable transcendental function *implicitly defined from Pfaffian functions* (or *existentially definable in* $\mathbb{R}_{\mathit{Pfaff}}$).
- Bound of (1) holds for S with $\dim(S) = 2$ IF S is implicitly defined from Pfaffian functions AND has a geometric property called *mild parameterization* (e.g. S definable in $\mathbb{R}_{\text{resPfaff}}$).