MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Alex Kruckman **Email/Phone:** Kruckman@gmail.com

Speaker's Name: Martin Hils

Talk Title: A Model Theoretic Approach to Berkovich Spaces (II)

Date: 02/06/14 **Time:** 9:30 am / pm (circle one)

List 6-12 key words for the talk: Hrushovski-Loeser spaces, stably dominated types, pro-definable sets, Berkovich spaces

Please summarize the lecture in 5 or fewer sentences: Part 2 of 3. The construction of Hrushovski and Loeser's "version" of Berkovich spaces: the space of stably dominated types on an algebraic variety. This space is not literally definable - it requires the machinery of pro-definable sets. The definable topology on Hrushovski-Loeser spaces, and limits of types.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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The Hrushovski-Loeser space associated to V ← algebraic variety

Thm (HHM) Let p be a definable type in ACVF. TFAE:
(1) $p \perp \Gamma$
(2) p is generically stable $(p(x) \otimes p(y) = p(y) \otimes p(x))$
(3) p is stably dominated

Note: $(1) \Rightarrow (3)$ is the interesting part. $(3) \Rightarrow (2) \Rightarrow (1)$ is easy.

Ex: $p_{\mathcal{O}}$ is "dominated" by $\mathrm{res}: \mathcal{O} \to k$ ← stable
Let $K \subseteq L$, $a \models p_{\mathcal{O}}|K$.
Then $a \models p_{\mathcal{O}}|L \Leftrightarrow \mathrm{res}(a) \notin k_L^{alg}$
$\Leftrightarrow \mathrm{res}(a) \underset{k_K}{\perp} k_L$

Fact: (1) If $p \in \widehat{X}(A)$, $f: X \to Y$ A-def., then $f_* p \in \widehat{Y}(A)$.
(2) If $tp(\bar{a}/K)$ is stably dominated and $\bar{b} \in acl(K \cup \{\bar{a}\})$, then $tp(\bar{b}/K)$ is stably dominated.
(3) If both $tp(\bar{a}/A)$ and $tp(\bar{b}/A \cup \{\bar{a}\})$ are stably dominated, then $tp(\bar{a}\bar{b}/A)$ is stably dominated.

Cor: (1) Let C be an alg. curve $/K \models ACVF$. Let $\bar{a} \in C(\mathcal{U})$.
Then $tp(\bar{a}/K) \in \widehat{C}(K) \Leftrightarrow td(K(\bar{a})/K) = td(k_{K(\bar{a})}/k_K)$.
[This uses: $\widehat{\mathbb{A}^1} = B^{cl}$, the set of closed balls]
(2) More generally, let $\bar{a}$ be any tuple in $\mathcal{U}^n$. If $td(K(\bar{a})/K) = td(k_{K(\bar{a})}/k_K)$, then $tp(\bar{a}/K)$ is stably dominated. The converse is false in general!

Goal: Put $\widehat{X}$ into the definable category.

Ex: • $\widehat{\mathbb{A}^1} = B^{cl}$ (def. set in $ACVF_{\mathcal{G}}$)
• Let $X \subseteq^{def} \Gamma_\infty^n$. Then $\widehat{X}(A) = X(A) \rightsquigarrow \widehat{X}$ is def. naturally.
In general, we need pro-definable sets.

<u>Def</u>: Let $W = K^N$ be a finite dim. vectorspace / $K \models ACVF$.
An $\mathcal{O}$-submodule $\Lambda \leq W$ is a semi-lattice if $\Lambda \simeq K^m \times \mathcal{O}^{N-m}$ for some m.

<u>Easy Fact</u>: The set $L(W)$ of semi-lattices in W is a definable family.

<u>Lemma</u>: A def. $\mathcal{O}$-submodule $\Lambda \leq W$ is a semi-lattice iff $\forall w \in W, w \neq 0$, either $K \cdot w \subseteq \Lambda$ or $K \cdot w \cap \Lambda \simeq \mathcal{O}$ (linear algebra).

Look at $\widehat{\mathbb{A}^n}$. For $d \geq 1$, let $W_d = K[X_1, \dots, X_n]_{<d}$. Let $p \in \widehat{\mathbb{A}^n}(K)$.
Define $\Lambda_d(p) = \{F \in W_d \mid \mathrm{val}(F(p)) \geq 0\}$.
<u>Note</u>: Clear by the Lemma: $\Lambda_d(p) \in L(W_d)$.

<u>Note</u>: The sequence $(\Lambda_d(p))_{d \geq 1}$ determines p completely.

<u>Def</u>: A <u>pro-definable set</u> is a projective limit $D = \varprojlim D_i(\mathcal{U})$, where D_i are definable and the transition functions $\pi_{ij} : D_i \to D_j$ are definable. Identify $D(\mathcal{U})$ with the correct subset of $\prod_{i \in I} D_i(\mathcal{U})$.

* There is a notion of a pro-definable map $f : D \to E$ for D and E pro-definable.
* A pro-def. set is <u>strict pro-definable</u> if it may be given by a surjective system of D_i's.
* If D is pro-def, $D = \varprojlim D_i$, $X \subseteq D$ is <u>relatively definable</u> if there is i and $X_i \subseteq^{\text{def}} D_i$ s.t. $X = \pi_i^{-1}(X_i)$.

<u>Thm</u>: Let X be a C-def set in ACVF. Then there is a strict pro-def. set (over C) $D = \varprojlim D_i$ s.t. $\forall A \supseteq C$, $D(A) = \widehat{X}(A)$ naturally.
<u>Remark</u>: Once established, we'll identify $\widehat{X}$ with D.

<u>Pf</u>: <u>$X = \mathbb{A}^n$</u>: For $d \geq 1$, let $E_d = L(W_d)$. $\pi_{d+1,d} : E_{d+1} \to E_d$.
Then $E = \varprojlim E_d$
$\quad\quad \rotatebox{90}{$\cong$}$
$D(\mathcal{U}) = \{(\Lambda_d(p))_d \mid p \in \widehat{\mathbb{A}^n}(\mathcal{U})\}$

"Same" objects as in the Berkovich construction (seminorms), but in a less naive way.

Need to show: $D_i := \pi_i(D)$

(1) $D(\mathcal{U}) = \varprojlim D_i$ $\Big\} \Leftarrow$ compactness +
(2) D_i is type-definable $\Big\}$ $p \in \widehat{X}$ iff $p \perp \Gamma$
(3) D_i is a union of def. sets $\} \Leftarrow$ $p \in \widehat{X}$ iff p is stably dominated
(2) + (3) + compactness $\Rightarrow$ result (D_i are definable)

General X: Take relatively definable subsets and glue.
Fact: If $X \subseteq^{def} Y$, then $\widehat{X} \subseteq \widehat{Y}$ is relatively definable.

A subset $X \subseteq \varprojlim D_i$ is called iso-definable if it is isomorphic to a def. set.

Remark: $X(\mathcal{U}) \subseteq \widehat{X}(\mathcal{U})$ is relatively definable and iso-definable.

Question: Are all iso-def. sets relatively def.?
Answer: Usually, the contrary! The notions are somehow orthogonal.

Thm: If C is an alg. curve, then $\widehat{C}$ is def.
Pf: We may assume C is smooth and projective, $C \hookrightarrow \mathbb{P}^n$, genus$(C) = g$.

Riemann-Roch $\Rightarrow$
- every $F \in K(C)^\times$ is a product of functions with $\leq g+1$ poles.
- $\exists N \in \mathbb{N}$ s.t. any $F \in K(C)^\times$ with $\leq g+1$ poles is given by $\left(\frac{G}{H}\right)\big|_C$, where $G, H \in K[X_0, \dots, X_n]_{\leq N}$.

Thus, enough to check on a definable family of rational functions.

Ex: $\widehat{\mathcal{O}^2}$ is not definable.
Consider $p_n(x,y) = p_{\mathcal{O}}(x) \cup \{y = x^n\}$. If $F(x,y)$ is of degree $< n$, then $val(F(p_n)) = val(F(p_{n+1}))$.
$\Rightarrow$ The types $(p_n)_{n \in \mathbb{N}}$ are not separated by a def. family of polynomials.

Remark: If $X \subseteq^{def} K^n$, then $\widehat{X}$ is def. iff $\dim_{Zar}(X^{Zar}) \leq 1$.

← in the case of a variety V

A def. topology on $\widehat{X}$ (on $\widehat{V}$):

* $\widehat{\mathbb{A}^n}$: The topology is generated by sets of the form $\{a\in\widehat{\mathbb{A}^n} \mid \mathrm{val}(F(a))>\gamma / <\gamma\}$, where $F\in\mathcal{U}[\bar{X}]$ and $\gamma\in\Gamma_\infty(\mathcal{U})$.

  Thus, we have countably many definable families of (relatively) def. basic open sets. Such a space is called a <u>(pro-)def. space</u>

  <u>Note</u>: If $K\models ACVF$, $\widehat{\mathbb{A}^n}(K)$ together with the K-def. open sets is a top. space. <u>But</u> $\widehat{\mathbb{A}^n}(K)\subseteq\widehat{\mathbb{A}^n}(\mathcal{U})$ is not continuous.

  → Similar to the situation in o-minimality: The restriction of the order topology to a submodel might be discrete!

* On $\widehat{V}$, define the topology on affine charts and glue the topologies to get a topology on $\widehat{V}$.
* If $X\subseteq^{def} V$, take the subspace topology on $\widehat{X}$. Such an $\widehat{X}\subseteq\widehat{V}$ will be called <u>semialgebraic</u>.

<u>Easy Facts</u>: (1) $\widehat{V}$ has a basis of open semialgebraic sets.

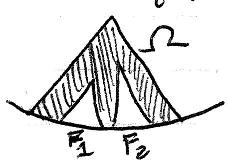
(2) $V(K)\hookrightarrow\widehat{V}(K)$ gets the valuation topology.

(3) $V(K)$ is dense in $\widehat{V}(K)$.

(4) $\widehat{V}(\mathcal{U})$ is Hausdorff.

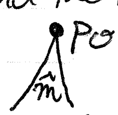
<u>Ex</u>: $V=\mathbb{A}^1$. A basis of definable open sets is given by sets of the form

$\widehat{\Omega\setminus\bigcup_{i=1}^{n}F_i}$, where Ω is an open ball and the F_i's are closed subballs.



$\widehat{\mathfrak{m}}$ is open.

$cl(\widehat{\mathfrak{m}})=\widehat{\mathfrak{m}}\cup\{p_{\mathcal{O}}\}$.



In particular, $int(\widehat{\mathcal{O}})=\bigcup_{x\in\mathcal{O}}\widehat{x+\mathfrak{m}}=\widehat{\mathcal{O}}\setminus\{p_{\mathcal{O}}\}$.

Example of an open non-semialgebraic set:

$\{p_B\in\widehat{\mathbb{A}^1} \mid \mathrm{radius}(B)>\gamma\}$, $\gamma<\infty$.

<u>Def</u>: $\widehat{V}$ is the <u>Hrushovski-Loeser space</u> (the <u>stable completion of V</u>).

<u>Def</u>: Let q be a def. type on a (pro-)def. space $(X,\mathcal{T})$. $a\in X(\mathcal{U})$ is a limit of q if $q(x)\vdash x\in\Omega$ for every def. open nbhd Ω of a.

<u>Ex:</u> $0^+ \in S_{def,\Gamma_\infty}$. Then $0 = \lim(0^+)$.

<u>Prop:</u> Let Z be a (pro-)def subset of $\widehat{V}$. Then the closure of Z is given by limits of definable types:

- If Z is closed, then any $q \in S_{def,Z}$ having a limit in $\widehat{V}$ has its limit in Z.
- $\forall a \in cl(Z)\ \exists q \in S_{def,Z}$ s.t. $\lim(q) = a$.

In the example, let $q \in S_{def,\widehat{m}}$. $q(x)$ says: x is the generic type of $B_{\geq \varepsilon}(0)$, with $\varepsilon \models 0^+$. $\lim(q) = p_{\mathcal{O}}$.

Example in the o-minimal setting (in Γ_∞):

$Z = \{(x,y) \in [0,\infty]^2 \mid x > 0, y < \infty\}$. $cl(Z) = [0,\infty]^2$.