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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Alex Kruckman **Email/Phone:** kruckman@gmail.com

Speaker's Name: Anand Pillay

Talk Title: Stability Theory and Diophantine Geometry (I)

Date: 02 / 06 / 14 **Time:** 11 : 00 [✓ am] / pm (circle one)

List 6-12 key words for the talk: Mordell-Lang, differentially closed fields, separably closed fields, Zilber trichotomy, stable groups,

Please summarize the lecture in 5 or fewer sentences: Part 1 of 2, Slides (pp. 1-60 of attached pdf) with supporting boardwork. An exposition of Hrushovski's proof of geometric Mordell-Lang. Ingredients: The model theory of differentially and separably closed fields, Zilber trichotomy in these theories (1-basedness and internality to the constants, Embedding Mordell-Lang data in the definable category, stable group theory.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
- **Overhead**: Obtain a copy or use the originals and scan them
- **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
- **Handouts**: Obtain copies of and scan all handouts

☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Semiabelian varieties:

semiabelian variety

short exact sequence
$$\begin{tikzcd} T \arrow[r] & G \arrow[r] & A \end{tikzcd}$$
torus

abelian variety

"Function-field" version of Mordell-Lang:

$$\begin{tikzcd} K \arrow[d, dash] \\ k \ (\text{e.g. } \overline{\mathbb{Q}}) \end{tikzcd}$$

Mordell-Lang is true as long as no data descends to k

$\mathcal{X}$ family of complex algebraic curves

$\uparrow\uparrow$ K-rational pts. of X = sections of the map $X \to S$

$S = \mathbb{P}^1(\mathbb{C})$ - finitely many points

Translating: $G \longrightarrow G_0/k$ adding a point

$X \longrightarrow X_0/k$

Cheap version: $(G, X) \longrightarrow (G_0, X_0)$ morphism, not necessarily isomorphism

"iso-trivial"?

$$\begin{tikzcd} \boxed{DF_{0,m}} \arrow[r, bend left, "\text{model}", "\text{companion}"'] & \boxed{DCF_{0,m}} \end{tikzcd}$$

DF_0, DCF_0 when $m = 1$.

No natural models! Standard function fields are very far from differentially closed.

Hasse derivations in $SCF_{p,m}$

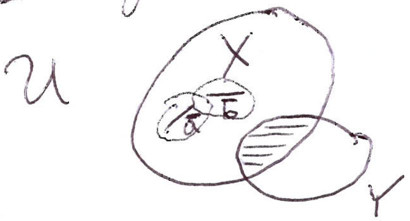
$\partial_i = (\partial_{i,n})_{n=1,2,\ldots}$ Convenient alternative to λ functions as in Pierre's talk.

---

type-definable = ctble intersection of def. sets. (might have to look in a saturated model to see the points)

relatively definable set = intersection of a definable set with the type-definable set.

Defining the terms in Zilber's Principle



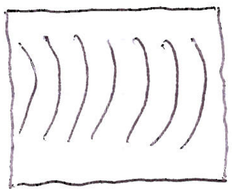
X type-definable
$Y \cap X$ relatively definable
X minimal means $Y \cap X$ or $X \setminus Y$ finite $\forall Y$.

X one-based means $\bar a \underset{\mathrm{acl}(\bar a)\cap \mathrm{acl}(\bar b)}{\perp} \bar b \quad \forall \bar a, \bar b$.

Curve: $Y \subseteq X \times X$ rel. def, dim 1   one-based $\Rightarrow$ no definable families of curves of dim > 1.

nonorthogonal to a minimal field
$\Rightarrow \exists$ def. $h : X \hookrightarrow F^n$
(in this context)

can't have this!



OK

Kolchin-closed sets: $P(\bar x, \bar x', \ldots, \bar x^{(m)}) = 0$, P a polynomial.

Step II, char. 0, context: $k < K$, G, X, Γ, Stab(X) finite, etc.

$\mu : G(K) \longrightarrow (K,+)^?$, $\Gamma \xrightarrow{\mu} \Gamma_1 \subseteq \Gamma \otimes k$, pull back is H.

Kernel $G^\#$ has finite Morley rank. $\Gamma < H$, $X^\# = X \cap H$.

Step II, char. p $K < \mathcal{U}$, $G(\mathcal{U})$ is G in the monster model
$X^\# \subset G^\#$, finite U-rank now, instead of Morley rank.

$X^\# \subset H$ in char 0
$X^\# \subset G^\#$ in char p

$$\underset{\text{model theoretic socle}}{S(H)} = \underset{\text{gen. by 1-based}}{\boxed{H_1}} + \underset{\text{gen. by non-1-based}}{H_2} \Longrightarrow H_1 \text{ 1-based}$$

Stability and diophantine geometry
MSRI introductory workshop

Anand Pillay

University of Notre Dame

February 8, 2014

Introduction

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- I will not discuss explicitly the stability-theoretic approach to Manin-Mumford over number fields, although this has also had an impact on current developments (e.g. algebraic dynamics).
- I will try to convey something of the richness of the mathematics, although the stability-theoretic background and tools are less accessible to the "outsider" (or even "insider") than o-minimality.

Statements I

- The origins are (i) the Mordell conjecture that a curve X of genus > 1 over a number field F has only finitely many F-rational points, and (ii) a conjecture of Manin that a curve of genus > 1 embedded in its Jacobian variety $J(X)$ meets only finitely many torsion points of $J(X)$.

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- A big common generalization of (i) and (ii) is Mordell-Lang: (char. 0.) If G is a semiabelian variety, $\Gamma < G$ is "finite-rank", i.e. contained in the division points of a finitely generated subgroup Γ_0, X a subvariety of G and $X \cap \Gamma$ is Zariski-dense in X, then X is a translate of an algebraic subgroup of G. When Γ is the just the torsion subgroup of G, this is sometimes called Manin-Mumford.

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- M-L was proved by Faltings, McQuillan... M-M is "easier", was first proved by Raynaud and has many other proofs, including by Hrushovski, and more recently Pila-Zannier (both with model theoretic input).

Statements II

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- It is worthwhile giving the "geometric" description of the latter when $K = \mathbb{C}(t)$, as this is often how things are described in the literature.
- X is the general fibre of a family of complex algebraic curves $\mathcal{X} \to S$ where S is $\mathbb{P}^1(\mathbb{C})$ minus finitely many points, X not defined over $\mathbb{C}$ means the family is nonconstant, and K-rational points of X are precisely rational sections $S \to \mathcal{X}$.

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- **Function field ML in characteristic** p, as formulated by Abramovich and Voloch, is just as above, but the "finite rank" assumption on Γ is replaced by: Γ is contained in the prime-to-p division points of a finitely generated subgroup Γ_0 of $G(K)$. (And defined over k is meant in a weaker sense.)

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- So we consider proofs of the above two statements, where the characteristic p case is the truly new theorem.

Differentially closed fields

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- Differential algebra already played a role in work of Manin and of Buium on characteristic 0 function field ML.

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- Again $\mathcal{C}$ is an algebraically closed field with "no induced structure" (be careful as it is just type-definable).

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- Note that 1-basedness and "internality to ..." make sense for any type-definable X, whether or not X is minimal.
- Zilber actually formulated the principle in the special case when X is "strongly minimal" (namely minimal and definable).

Dichotomy II.

So Step I of the proof (connected with earlier work of Hrushovski, Zilber, Sokolovic, and of independent interest) is:

Theorem 0.1

The Zilber principle is true for $DCF_{0,m}$ *and* $SCF_{p,m}$ *(for finite-dimensional, also called thin, minimal types). Moreover in either case* F *can be taken to be the field of (absolute) constants* $\mathcal{C}$.

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- The proof goes via "Zariski geometries" (or structures). A Zariski geometry, as defined in HZ, or in Zilber's book is a strongly minimal set X (in an ambient structure if one wishes) such that certain subsets of $X, X \times X, ..$ are designated to be closed, and these closed sets satisfy abstract conditions somewhat like the Zariski closed sets of Cartesian powers of a smooth algebraic curve.

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- The HZ theorem is that the Zilber principle holds for Zariski geometries (and this theorem is the raison d'etre for the notion of Zariski geometries).

Dichotomy III

- All proofs of the HZ theorem are complicated to say the least, involving an abstract notion of tangency of closed definable sets as well as Hrushovski's group and field configuration theorem. I don't want to say anything more about it now but will discuss possible direct treatments in the case at hand in my second talk.

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- The proof that strongly minimal sets in $DCF_{0,m}$ are Zariski is straightforward, more or less taking the closed sets to be the Kolchin closed sets (and generalizes easily to arbitrary finite Morley rank sets in $DCF_{0,m}$ being "Zariski-type structures"). Together with a classification of definable fields in $DCF_{0,m}$ this yields Theorem 0.1 for $DCF_{0,m}$.

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- To prove Theorem 0.1 for $SCF_{p,m}$ by such methods requires dealing with *type-definable* Zariski geometries. These do not appear in Zilber's book and are not explicit in HZ. Nevertheless arguments are given for adapting HZ to this case and proving that thin, minimal sets in $SCF_{p,m}$ are Zariski.

Dichotomy IV

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Dichotomy IV

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- In any case, Theorem 0.1 is at the core of Hrushovski's approach to ML (although in our expositions 20 years ago we tended to take this as an unexplained black box).
- The remainder of the proof involves:
  Step II.: embedding the ML data into a definable framework (DCF_0/SCF_p),
  Step III. using stable-group-theoretic arguments together with Step I (Theorem 0.1) to obtain descent of the (type)-definable data to the constants, and
  Step IV.: deducing descent of the original algebraic geometric data.

Step II, characteristic 0

- So we have the data k, K, G, X, Γ. G,X, Γ are defined over the algebraic closure of $K_0 = k(t_1, .., t_m)$ for some m. Put the canonical partial differential structure on K_0 and pass to the differential closure which we may assume to be K and has constants k. (In fact we may take $m = 1$.)

Step II, characteristic 0

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- Buium defines a generalized "logarithmic derivative" homomorphism μ (definable in the differential field K) from $G = G(K)$ to some $(K, +)^n$, whose kernel $G^\sharp$ has finite Morley rank.

Step II, characteristic 0

- So we have the data k, K, G, X, Γ. G,X, Γ are defined over the algebraic closure of $K_0 = k(t_1, .., t_m)$ for some m. Put the canonical partial differential structure on K_0 and pass to the differential closure which we may assume to be K and has constants k. (In fact we may take $m = 1$.)
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- The original data G, X, Γ has been replaced by the definable data G, X, H. Moreover $X^\sharp$ has trivial stabilizer in H.

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- Γ has been replaced by the type-definable group $G^\sharp$, which can be shown to be finite-dimensional, so finite U-rank (but not necessarily Morley rank).

Stable group theory 1

- We start with **Step III** in the char. 0 case, namely we have the finite Morley rank (commutative) group H definable in $K \models DCF_0$ and $X^\sharp$ is a definable subset of H with trivial stabilizer.

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- Udi proves the (weak) socle theorem for arbitrary finite Morley rank groups H (nice but not too hard)

Theorem 0.2

Suppose H is sufficiently "rigid" in the sense of having no infinite definable families of definable subgroups. Suppose Y is a definable subset of H with finite stabilizer. Then, up to to translation $Y \subseteq s(H)$.

Stable group theory 2

- Now, in general we can write $s(H) = H_1 + H_2$ where H_1 is generated by 1-based strongly minimal sets, and H_2 is generated by non 1-based strongly minimal sets. It follows that H_1 is 1-based in its own right and early work (HP) yields:

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- By Theorem 0.2, after a translation $X^\sharp$ is contained in $s(H)$.
- The **structure of 1-based groups** above and the assumption that $Stab(X^\sharp)$ is trivial implies that after a further translation $X^\sharp \subseteq H_2$.

Stable group theory 3

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- Step IV is obtained by taking Zariski closures of the (type) definable data. This involves additional arguments, especially in the characteristic p case where we want to deduce descent to k from descent to $\mathcal{C}$. But more or less straightforward. End of outline.

Although not made explicit in the sketch above, a key ingredient is:

Theorem 0.3

Suppose A is an abelian variety over $\mathcal{U}$ with $\mathcal{C}$-trace 0 then $A^\sharp$ is 1-based, and moreover (strongly) minimal when A is simple.

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- The "bad news" is that the approach does not always work in positive characteristic case, and only recovers Mordell-Lang for so-called ordinary semiabelian varieties, already done by Abramovich-Voloch.
- So in the first part of this second talk I will give a few details.

Differential jet spaces 1

- Suppose first that V is an (affine) algebraic variety over an algebraically closed field K, and $a \in V(K)$. We have the higher tangent spaces of V at a, namely $j_n(V)_a$ is the dual space to m/m^{n+1} where m is the maximal ideal of V at a namely the set of functions in the coordinate ring $K[V]$ of V, which vanish at a.

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- Exactly the same thing holds for "finite-dimensional" differential algebraic varieties, in characteristic 0 at least, which we will discuss next.

Differential jet spaces 2

- If X is defined by $f(y, y') = 0$ say then the Kolchin tangent space to X at a point a is defined by the linear differential equation $(\partial f/\partial y)(a)(u) + (\partial f/\partial y')(a)(u') = 0$.

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- For a differential algebraic variety X and $a \in X$ the higher Kolchin tangent spaces $j_n^{\partial}(X)_a$ (differential jet spaces) at $a \in X$ are defined by linear differential equations, and "finite-dimensionality" of X corresponds to these spaces being finite-dimensional vector spaces over the constants.

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- Hence **Fact 1**: If $(Y_b : b \in Z)$ is a differential algebraic family of differential algebraic subvarieties of (finite-dimensional) X, all passing through $a \in X$, we have a differential rational (so definable) embedding of Z in $Gr(j_n^{\partial}(X)_a)$ for some n.

Differential jet spaces 3.

- Now suppose X is a strongly minimal (so finite-dimensional) differential algebraic variety. Non 1-basedness of of X means precisely that there is an infinite definable family $(Y_b : b \in Z)$ of differential algebraic subvarieties of $X \times X$ passing through a generic point $a \in X \times X$.

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- Now $j^n(X \times X)_a$ is internal to $\mathcal{C}$, hence by Fact 1, so is Z. This yields some definable relationship between X and $\mathcal{C}$ which is enough to prove Theorem 0.1 for DCF_0.

In fact we also obtain:

Theorem 0.4

(Strong socle theorem) Let G be a finite Morley rank group in DCF_0 and Y a differential algebraic subvariety with trivial stabilizer. Then Y is internal to $\mathcal{C}$.

Differential jet spaces 4.

- The proof is: $(Y_b = Y - b : b \in Y)$ is a canonical definable family of differential algebraic subvarieties of G passing through 0, hence Y definably embeds in some $Gr(j_n^{\partial}(G)_0)$ so is internal to $\mathcal{C}$.
- This yields directly Step III of the proof in the characteristic 0 case.

Positive characteristic case 1

- The situation described above depends essentially on being able to describe a finite-dimensional differential algebraic variety in characteristic 0 (up to a change of coordinates) as the solution set of a first order polynomial differential equation $\partial(y) = s(y)$ on an algebraic variety V (so s is a kind of Ehresmann connection on V).

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- So in positive characteristic ($SCF_{p,1}$ say), the approach directly works for so-called ∂-varieties X, where ∂ is the iterative Hasse derivation.
- Namely sets X of the form $\{x \in V(\mathcal{U}) : \partial_n(x) = s_i(x) : i = 1, 2, ...\}$ for V an algebraic variety and s_n suitable polynomial functions.

Positive characteristic case 2

- So in this case the differential tangent space (for example) at a good point is defined by an iterative Hasse linear differential system: $\{\partial_n(y) = A(y) : n = 1, 2, ...\}$, whose solution set is a finite-dimensional vector space over $\mathcal{C}$.

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- The approach also works for type-definable definable sets X such that for generic $a \in X$, $\partial_n(a)$ is separably algebraic over a for eventually all n (so called very thin types). And for finite-dimensional groups whose generic type is very thin, we also obtain the Strong socle theorem.

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- It remains open to find a transparent jet-space account of Theorem 0.1 and/or Theorem 0.3 for traceless abelian varieties A in the positive characteristic case. See later.

Miscellanea

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- Moosa and Scanlon have substantially generalized the jet space arguments to fields with operators.
- Also Theorem 0.3 was used (together with other ingredients) to obtain an Ax-Lindemann theorem for nonconstant semiabelian varieties. (BP) The work is ongoing.

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- In so far as it works it also gives deductions from MM of Theorem 0.3 for example.
- Note that such an elementary strategy could not work in the absolute case, where MM and ML are of different orders of difficulty.

MM implies ML II

- It is convenient to take the contrapositive of the contrapositive in the statement of ML, with a slightly stronger hypothesis and conclusion:
- **Function field ML: restatement** Let $K = \mathbb{C}(t)^{alg}$ in char. 0, and $= \mathbb{F}_p(t)^{sep}$ in char. p and k be the "constants", $\mathbb{C}$, $\mathbb{F}_p^{alg}$ respectively. Let A be an abelian variety over K with k-trace 0. Let X be an irreducible subvariety of G (defined over K), $\Gamma \subset G(K)$ be as before, namely (prime-to-p) division points of a finitely generated subgroup of G, and assume $X \cap \Gamma$ is Zariski-dense in X. THEN X is a translate of an abelian subgroup of G (by a point of Γ). Now the MM statement is when Γ is contained in the group of all torsion points of G.

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- So the **Basic Strategy** is: MM + Theorem of the kernel + Frank implies ML (and also Theorem 0.3).

Theorem of the kernel

- $A^\sharp$ can be defined as the smallest Zariski-dense (type)-definable subgroup of A, where in the positive characteristic case we read this in a saturated model, but in any case in positive char. case $A^\sharp(K) = \cap_n p^n(A(K))$ and can also be described as the maximal divisible subgroup of $A(K)$.

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- In positive characteristic the statement was recently proved by Roessler.

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Theorem 0.5

(Frank) Suppose G is g-minimal. Then any infinite algebraically closed subset of G is an elementary substructure.

So g-minimal groups behave somewhat like strongly minimal sets. The result was originally proved by Frank for arbitrary fields of finite Morley rank, with relevance to "bad groups".

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- By the weak socle theorem we may assume that $X^\sharp$ is contained in $s(H)$.

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- As K^{diff} is the prime model over K it follows that $A^\sharp(K^{diff}) = A^\sharp(K)$ which by the Theorem of the kernel is precisely the torsion points of A.
- By Manin-Mumford and (*), X is a translate of an abelian subvariety of A. End of proof and/or contradiction.

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- Not always true, but I know no example of a type-definable minimal group which does not have QE.

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- By hypothesis $A^\sharp$, as a structure in its own right has QE, and hence has finite Morley rank, and is a sum of g-minimal definable subgroups.

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- By the Theorem of the kernel $A^\sharp(K)$ consists of torsion points, so by Manin-Mumford (proved by Pink-Roessler), X is a translate of an abelian subvariety of A. End of proof of Theorem 0.6.